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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

... ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

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Hypo-Elasticity and Elasticity

BARRY BERNSTEIN

Communicated by C. TRUESDELL

1. Introduction

When TRUESDELL [1955, 6] introduced the equations of hypo-elasticity, he stated that it was his intention to find a new concept of elastic behavior, mutually exclusive with the theory of finite strain, which reduces to the classical linearized theory of elasticity under the assumptions appropriate to the latter. However, he remarked that NOLL had shown him privately, and published simultaneously [1955, 2] a proof that every isotropic Cauchy-elastic material for which the stress-strain relations are invertible is hypo-elastic.

A question then arises whether a given hypo-elastic material is elastic. And this inquiry is the theme of our discussion below. In seeking to answer this question I was impeded by a difficulty: The equations of hypo-elasticity in themselves were not sufficient to define a material well enough to enable me to proceed with my inquiry. This hurdle was overcome by the means described in § 3 below, in which are set down the stipulations which, in addition to the equations of hypo-elasticity, I feel are required in general to define a hypo-elastic material.

In the second section we introduce our notation for discussing motions of materials and define elasticity in the sense of CAUCHY and of GREEN. In the fourth and fifth sections we derive criteria, based on the equations of hypo-elasticity, for a hypo-elastic material to be elastic in the senses of CAUCHY and of GREEN. These criteria are obtained by means of seeking integrability conditions.

The sixth section is devoted to showing that unless there exist motions of a given hypo-elastic material for which the initial and final configurations are the same and the work done by the stresses is negative, the given material must necessarily be elastic in the sense of GREEN. The existence of a result of this type was suggested by the work of CAPRIOLI [1955, 1] and of BERNSTEIN & ERICKSEN [1958, 1].

In the final section, § 7, we prove that a material which is both hypo-elastic and Cauchy-elastic is isotropic with respect to a given reference configuration if and only if the stress is a uniform hydrostatic pressure for that reference configuration. We then indicate that this result implies that there exist anisotropic Green-elastic materials which are not hypo-elastic.

A summary of our results is given by BERNSTEIN [1960].

2. The constitutive equations of elasticity

We follow generally the notation of TRUESDELL [1952], letting X^A, $A=1, 2, 3$, denote material or Lagrangean coordinates, x^i, $i=1, 2, 3$, stand for spatial or Eulerian coordinates, and t for time. A (kinematically possible) *motion* is defined by a continuous, piecewise smooth mapping

$$x^i = \hat{x}^i(X^A, t), \tag{2.1}$$

where carats, bars, *etc.*, over a letter indicate that it stands for a function, a distinction which will be made when the context requires it. Marks appearing under or to the right of a kernel letter will be taken to be part of the kernel letter. Henceforth all spatial coordinates will be considered cartesian. Components of tensors in spatial coordinate systems will be written with lower case Latin kernel letters and lower case Latin indices, while those in material coordinate systems will be written with upper case Latin kernel letters and upper case Latin indices. Since all spatial coordinate systems are to be cartesian, cartesian tensor notation will be used in these systems, and lower case Latin indices may be transferred between the upper and lower positions at will. A *material point* will be denoted by X, $\underset{0}{X}$, *etc.*, when no reference is made to coordinates.

The derivatives of (2.1) with respect to the X^A, called the *displacement gradients*, will be written

$$x^i_A = \frac{\partial \hat{x}^i}{\partial X^A}. \tag{2.2}$$

It is assumed that (2.1) may be solved for the X^A, yielding

$$X^A = \hat{X}^A(x^i, t), \tag{2.3}$$

which leads to the definition of the *inverse displacement gradients*, X^A_i by

$$X^A_i = \frac{\partial \hat{X}^A}{\partial x^i}, \tag{2.4}$$

and the subsequent relations

$$X^A_i\, x^i_B = \delta^A_B, \qquad x^i_A\, X^A_j = \delta^i_j, \qquad x^i_A = X^A_i \det|x^j_L|. \tag{2.5}$$

By differentiation of (2.5) we get

$$\frac{\partial}{\partial x^j_A} \det|x^i_B| = X^A_j \det|x^i_B|, \qquad \frac{\partial X^A_i}{\partial x^j_L} = -X^A_j\, X^L_i. \tag{2.6}$$

When $\hat{x}^i_A$, $\tilde{x}^i_A$, *etc.*, denotes an assignment of the displacement gradients as functions of given variables, $\widehat{X}^A_i$, $\widetilde{X}^A_i$ *etc.* will respectively denote the corresponding inverse displacement gradients determined by (2.5) as functions of the same variables.

The symbols δ^i_j and δ^A_B appearing in (2.5) are the usual Kronecker deltas. We also adopt the Kronecker symbols δ^i_A and δ^A_i, which take the value unity when A and i are numerically equal, and take the value zero otherwise*.

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* See [1954], footnote on page 14.

The *mass density*, ϱ, is given from the principle of conservation of mass by

$$\varrho_0 \det |x^i_A| = \varrho, \tag{2.7}$$

where ϱ_0 depends only on X and in fact is the density for any value of x^i_A such that $\det |x^i_A| = 1$.

If at each point of a smooth manifold $\mathfrak{M}$ of material points* the stress, t^{ij}, is given as a function of the displacement gradients,

$$t^{ij} = \hat{t}^{ij}(x^i_A, X^B), \tag{2.8}$$

we say that we have a *representation* of a material which is *elastic in the sense of* CAUCHY, or *Cauchy-elastic*. These representations form equivalence classes if we say that, for fixed $\mathfrak{M}$, two representations

$$t^{ij} = \hat{t}^{ij}(x^k_A, X^B) \quad \text{and} \quad t^{ij} = \hat{\hat{t}}^{ij}(x^k_A, X^B)$$

are equivalent if there is a transformation of material coordinates

$$X^{B'} = \hat{X}^{B'}(X^B) \tag{2.9}$$

such that

$$\hat{t}^{ij}(x^k_A, X^B) = \hat{\hat{t}}^{ij}(x^k_{A'}, X^{B'}) \tag{2.10}$$

where

$$x^k_{A'} \equiv \frac{\partial \hat{x}^k}{\partial X^{A'}} = x^k_A \frac{\partial \hat{X}^A}{\partial X^{A'}}.$$

An equivalence class of representations of Cauchy-elastic materials will be called by us a *Cauchy-elastic material*. Such a material is well defined by one of its representations.

The possible relations (2.8) are restricted by the requirement that material properties be invariant under rigid motions. This requirement has been given the name of the principle of *isotropy of space*, or *objectivity* [1955, 2] [1958, 2]. It has also been formulated as *invariance under Euclidean transformations*, and it has been shown [1958, 3] that (2.8) will be consistent with this principle if and only if it takes the form

$$t^{ij} = [\det |x^k_M|]^{-1} x^i_A x^j_B \hat{T}^{AB}(C^{KL}, X^A), \tag{2.11}$$

where

$$C^{KL} \equiv X^K_i X^L_j \delta^{ij}, \tag{2.12}$$

and $\hat{T}^{AB}$ are scalar functions with respect to time-dependent orthogonal transformations of spatial coordinates.

In case (2.8) has the form

$$t^i_j = \frac{\varrho}{\varrho_0} x^A_j \frac{\partial \hat{\Sigma}}{\partial x^i_A}, \tag{2.13}$$

where

$$\hat{\Sigma}(x^i_A, X)$$

is, at each X, a scalar function of the displacement gradients, then we say that the material defined by $\mathfrak{M}$ and (2.13) is *elastic in the sense of* GREEN, or *Green-*

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* $\mathfrak{M}$ is the same as the body $\mathscr{B}$ of [1958, 2].

elastic. (If (2.8) has the form (2.13) for one representation, it has that form for all equivalent representations.)

The principle of isotropy of space is fulfilled, for materials elastic in the sense of GREEN, if and only if $\hat{\Sigma}$ is a function of the C^{KL} and X [1955, 2].

3. Hypo-elastic materials

The constitutive equations of hypo-elasticity may be written [1955, 2, 4, 5]

$$\dot{t}_{ij} = t_{ik}\,\omega_{jk} + t_{jk}\,\omega_{ik} + \hat{A}_{ijkl}(t_{pq})\,d_{kl}, \tag{3.1}$$

where, if a quantity f is expressed as a function of X and t, $f = \hat{f}(X, t)$, then $\dot{f} \equiv \partial\hat{f}/\partial t$ is called the material derivative of f; where

$$\omega_{jk} = \tfrac{1}{2}\,(v_{j,k} - v_{k,j}), \qquad d_{kl} = \tfrac{1}{2}\,(v_{k,l} + v_{l,k}), \tag{3.2}$$

v_i denoting the velocity, which is defined, using (2.1), by

$$v_i = \frac{\partial\hat{x}^i}{\partial t}; \tag{3.3}$$

and where $\hat{A}_{ijkl}(t_{pq})$ is an invariant or isotropic tensor function of the stress and possesses the symmetries $\hat{A}_{ijkl} = \hat{A}_{jikl} = \hat{A}_{jilk}$. Using (3.3), (2.1), (2.2) and (2.4), we may write the velocity gradients as

$$v^i{}_{,j} = X^A_j\,\dot{x}^i_A. \tag{3.4}$$

Substitution of (3.4) into (3.1), using (3.2), yields

$$\dot{t}_{ij} = \hat{B}_{ijkl}(t_{pq})\,X^A_l\,\dot{x}^k_A, \tag{3.5}$$

where

$$\hat{B}_{ijkl} = \tfrac{1}{2}\,[t_{il}\,\delta_{jk} + t_{jl}\,\delta_{ik} - t_{ik}\,\delta_{jl} - t_{jk}\,\delta_{il}] + \hat{A}_{ijkl}. \tag{3.6}$$

In the spirit of differential geometry, we shall assume that $\hat{A}_{ijkl}$, and hence $\hat{B}_{ijkl}$, is differentiable as many times as required, and that we have enough continuity conditions to insure existence and uniqueness of solutions to the initial-value problem for (3.1) or (3.6) in which the velocity gradients, or the displacement gradients, are assigned as continuous piecewise smooth functions of time, stress is assigned initially, and it is required to solve for stress as a function of time at each X. Our theorems are local unless obviously global.

We do not feel that equations (3.1) or (3.5) suffice to define a hypo-elastic material. The following example will bring out the difficulty which we have in mind.

Consider the particular case of (3.1) given by

$$\dot{t}_{ij} = t_{ik}\,\omega_{jk} + t_{jk}\,\omega_{ik} - \delta_{ij}\,\frac{t_{pp}}{3}\,d_{ll},$$

i.e.,

$$\hat{A}_{ijkl} = -\,\delta_{ij}\,\frac{t_{pp}}{3}\,\delta_{kl}. \tag{3.7}$$

One set of solutions to (3.7) is given by

$$t_{ij} = -\,\varkappa\,\varrho\,\delta_{ij}, \tag{3.8}$$

where $\varkappa$ is an arbitrary constant. Hence if initially the stress is a hydrostatic pressure with a given assigned value, $\varkappa$ may be adjusted so that the right-hand side of (3.8) takes that value initially and, by uniqueness, (3.8) must be the solution. It follows that if ever the stress is a hydrostatic pressure, it is always a hydrostatic pressure, and if the stress is once not a hydrostatic pressure, it may never become a hydrostatic pressure during any motion. Thus it would seem that (3.7) entails at least two essentially different types of materials, one for which the shear must vanish and one for which the shear cannot vanish. And we shall see (in the closing comments of §§ 4 and 5) that this difference is even more profound.

What is clear from the foregoing discussion is that in some way the possible initial conditions must be specified. And we now proceed to formalize this idea.

Let $\mathfrak{M}$ be a smooth manifold of material points as in § 2. A function $\hat{x}^i_A(X^B)$ over $\mathfrak{M}$, *i.e.*, an assignment of displacement gradients over $\mathfrak{M}$, subject to the compatibility conditions

$$\frac{\partial \hat{x}^i_A}{\partial X^B} = \frac{\partial \hat{x}^i_B}{\partial X^A}, \tag{3.9}$$

will be called a *configuration* (of $\mathfrak{M}$). An assignment of both a value of stress at each material point in $\mathfrak{M}$ and a configuration will be called a *stress-configuration pair* and will be denoted by

$$\{t_{ij}, x^k_A\}. \tag{3.10}$$

Now (3.10) is a function over $\mathfrak{M}$. Its value at one point X of $\mathfrak{M}$ will be denoted by

$$[t_{ij}, x^k_A]\,(X). \tag{3.11}$$

The symbol

$$[t_{ij}, x^k_A] \tag{3.12}$$

will denote a possible set of values of stress and displacement gradients at a material point. It will be called a *stress-displacement gradient pair.* The expression (3.11) is necessarily associated with a given stress-configuration pair, while (3.12) is not.

Given two stress-configuration pairs

$$\{\overset{1}{t}_{ij}, \overset{1}{x}{}^k_A\} \quad \text{and} \quad \{\overset{2}{t}_{ij}, \overset{2}{x}{}^k_A\},$$

the following statement may or may not be true: There exist continuous piecewise smooth functions

$$\hat{t}_{ij}(t, X), \qquad \hat{x}^k_A(t, X) \qquad X \in \mathfrak{M}, \qquad t_1 \leqq t \leqq t_2$$

satisfying (3.5) and (3.9) such that

$$[\hat{t}_{ij}(t_l, X), \hat{x}^k_A(t_l, X)] = [\overset{l}{t}_{ij}, \overset{l}{x}{}^k_A]\,(X), \qquad l = 1, 2$$

for all $X \in \mathfrak{M}$.

If this statement is true, we shall say that the two stress-configuration pairs are equivalent,

$$\{\overset{1}{t}_{ij}, \overset{1}{x}{}^k_A\} \sim \{\overset{2}{t}_{ij}, \overset{2}{x}{}^k_A\},$$

and if not, then the stress-configuration pairs are not equivalent. It is readily verified that this is an equivalence relation. And an equivalence class of stress-configuration pairs will be called a *stress-configuration class.* A stress-configuration class for a hypo-elastic material is, then, the class of all stress-configuration pairs which may be reached from a given stress-configuration pair. Further, these stress-configuration classes themselves may be separated into equivalence classes as follows: Two stress-configuration classes are equivalent if one may be obtained from the other by a transformation of material coordinates.

In other words, given two stress-configuration classes, say Γ_1 and Γ_2, Γ_1 is equivalent to Γ_2 if there exists a coordinate transformation

$$X^{\bar{A}} = \bar{X}^{\bar{A}}(X^A) \tag{3.13}$$

such that if

$$\{t_{ij}, x^i_A\} \in \Gamma_1,$$

then

$$\{t_{ij}, x^i_{\bar{A}}\} \in \Gamma_2$$

and conversely, where

$$x^i_A = x^i_{\bar{A}} \frac{\partial \bar{X}^{\bar{A}}}{\partial X^A}.$$

From equation (3.5) it is readily verified that

$$\dot{t}_{ij} - B_{ijkl}(t_{pq})\, X^A_l\, x^k_A = \dot{t}_{ij} - B_{ijkl}(t_{pq})\, X^{\bar{A}}_l\, x^k_{\bar{A}}, \tag{3.14}$$

where

$$X^{\bar{A}}_l = \frac{\partial \bar{X}^{\bar{A}}}{\partial x^l} = \frac{\partial \bar{X}^{\bar{A}}}{\partial X^A}\, X^A_l.$$

Thus if any stress-configuration pair from Γ_1 may be transformed into a stress-configuration pair from Γ_2 by a transformation (3.12), it is immediate from (3.13) that every stress-configuration pair in Γ_2 is a transform under (3.12) of a stress-configuration pair in Γ_1, and conversely. Hence, necessary and sufficient that Γ_1 and Γ_2 be equivalent is that one stress-configuration pair in Γ_1 may be transformed into some stress-configuration pair in Γ_2 by a transformation of material coordinates.

By a hypo-elastic material we shall mean the assignment of a set of equations (3.1) *or* (3.5) *and a corresponding equivalence class of stress-configuration classes. By a representation of a hypo-elastic material we shall mean an assignment of a set of equations* (3.1) *or* (3.5) *and a corresponding stress-configuration class.*

It may be noted that in a representation of a hypo-elastic material which is also Cauchy-elastic there is only one value of stress for each value of the displacement gradients at each X, and hence a representation of such a material consists of an assignment of stress as a function of x^i_A and X. Also the equivalence classes of stress-configuration classes must be those obtained by considering classes of equivalent representations of an elastic material. Hence our definitions of a representation and a material are consistent for the case of a material which is both Cauchy-elastic and hypo-elastic.

4. Elasticity in the sense of Cauchy

Suppose that we are given a set of hypo-elastic equations (3.1) or (3.5), and we ask whether any hypo-elastic material corresponding to them is elastic in the sense of CAUCHY. If (2.8) defines such a material, then by substitution of (2.8) into (3.5) we see that $\hat{t}_{ij}$ must satisfy

$$\frac{\partial \hat{t}_{ij}}{\partial x^k_A} \dot{x}^k_A = \hat{B}_{ijkl} X^A_l \dot{x}^k_A \tag{4.1}$$

for arbitrary motions. But since x^k_A and $\dot{x}^k_A$ are in general independent, (2.8) must satisfy (4.1) for arbitrary $\dot{x}^k_A$ and hence must satisfy

$$\frac{\partial \hat{t}_{ij}}{\partial x^k_A} = \hat{B}_{ijkl} X^A_l . \tag{4.2}$$

Conversely, if we have a solution (2.8) of (4.2), then (4.1) and hence (3.5) is satisfied by this solution and, because of uniqueness of solution of the initial value problem for (3.5) as stated in § 3, the stress-configuration class determined by any single stress-configuration pair satisfying (2.8) must be identical with the totality of stress-configuration pairs satisfying (2.8). And thus this stress-configuration class, together with (3.5), determines a representation of a material which is both hypo-elastic and Cauchy-elastic. Thus our question reduces to that of whether (4.2) admits a solution. We therefore apply the standard method of answering this question as outlined in textbooks on differential geometry*.

The first integrability condition for (4.2) is obtained by writing

$$\begin{aligned} 0 &= \frac{\partial^2 \hat{t}_{ij}}{\partial x^p_L \, \partial x^k_A} - \frac{\partial^2 \hat{t}_{ij}}{\partial x^k_A \, \partial x^p_L} \\ &= \frac{\partial \hat{B}_{ijkl}}{\partial t_{rs}} \frac{\partial \hat{t}_{rs}}{\partial x^p_L} X^A_l + \hat{B}_{ijkl} \frac{\partial X^A_l}{\partial x^p_L} - \frac{\partial \hat{B}_{ijpl}}{\partial t_{rs}} \frac{\partial \hat{t}_{rs}}{\partial x^k_A} X^L_l - \hat{B}_{ijpl} \frac{\partial X^L_l}{\partial x^k_A} . \end{aligned} \tag{4.3}$$

By substituting (4.2) for the derivatives of $\hat{t}_{rs}$ and using $(2.6)_2$, we get a set of relations involving X^A_k and t_{pq}. From these X^A_k may be eliminated using $(2.5)_2$, so that the first integrability conditions for (4.2) take the form

$$\frac{\partial \hat{B}_{ijkl}}{\partial t_{rs}} \hat{B}_{rspq} - \frac{\partial \hat{B}_{ijpq}}{\partial t_{rs}} \hat{B}_{rskl} - \hat{B}_{ijkq} \, \delta_{pl} + \hat{B}_{ijpl} \, \delta_{kq} = 0 . \tag{4.4}$$

For t_{ij} to satisfy (4.2) it is necessary that its components be related in such a way as to satisfy (4.4). If (4.4) is not satisfied identically in t_{ij} and does not consist of more than six independent relations, we may differentiate (4.4) with respect to x^k_A, substitute the right-hand side of (4.2) for the derivatives of t_{ij}, and again eliminate X^A_k by the use of $(2.5)_2$ to obtain the second integrability conditions as relations among the t_{ij}. If the second integrability conditions introduce new relations not contained among the first integrability conditions, and if the total number of independent relations among the first and second integrability conditions does not exceed six, we obtain the third integrability conditions by applying to the second integrability conditions the same process that we applied to the first. And we continue this process till either we obtain more than six independent relations, or at some stage no new relations are introduced and the total number of independent relations is $r \leqq 6$. In the former case there is no solution of (4.2). In the latter case there are solutions for which $6-r$ components of t_{ij} may be assigned arbitrarily for one value of x^k_A at each X. And for each such assignment the solution is unique.

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* See, for example, § 23 of [1947].

In case (4.4) consists of more than six independent relations there is, according to the above discussion, no solution of (4.2). On the other hand if and only if (4.4) is satisfied identically in t_{ij} is there a solution of (4.2) for any arbitrary assignment of t_{ij} for one value of x^k_A at each X. Thus, because of the uniqueness of solutions to the initial value problem for (3.5), we have the following:

Theorem I. *Necessary and sufficient that every hypo-elastic material corresponding to* (3.1) *or* (3.5) *be Cauchy-elastic is that* (4.4) *be satisfied identically in* t_{ij}.

As an example to illustrate the foregoing discussion, we consider the equations (3.7), for which the totality of independent integrability conditions may be written

$$t_{ij} = \frac{t_{kk}}{3}\,\delta_{ij}, \tag{4.5}$$

showing that a hypo-elastic material corresponding to (3.7) is elastic in the sense of Cauchy if and only if the stress is a hydrostatic pressure.

5. Elasticity in the sense of Green

Given a material which is both hypo-elastic and Cauchy-elastic, we now ask for a criterion that it be elastic in the sense of Green. We ask, in other words, if there exists a function $\widehat{\Sigma}(x^i_A, X^B)$ satisfying (2.13). We rewrite (2.13) as

$$\frac{\partial \widehat{\Sigma}}{\partial x^i_A} = \frac{\varrho_0}{\varrho}\, t_{ij}\, X^A_j. \tag{5.1}$$

Since we are assuming that (2.8) is satisfied, a necessary and sufficient condition for the existence of a solution $\widehat{\Sigma}$ to (5.1) is

$$0 = \frac{\partial^2 \widehat{\Sigma}}{\partial x^k_L\, \partial x^i_A} - \frac{\partial^2 \widehat{\Sigma}}{\partial x^i_A\, \partial x^k_L} = \frac{\partial}{\partial x^k_L}\left[\frac{\varrho_0}{\varrho}\, t_{ij}\, X^A_j\right] - \frac{\partial}{\partial x^i_A}\left[\frac{\varrho_0}{\varrho}\, t_{kj}\, X^L_j\right]. \tag{5.2}$$

From (2.7) and (2.6) we get

$$\frac{\partial}{\partial x^i_A}\,\frac{\varrho_0}{\varrho} = \frac{\varrho_0}{\varrho}\, X^A_i. \tag{5.2}$$

Since we are also assuming that (3.5) is satisfied, we get from (5.1), using (5.2), (3.5) and $(2.6)_2$,

$$\widehat{B}_{ijpk}\, X^L_p\, X^A_j - \widehat{B}_{kjpi}\, X^A_p\, X^L_j + t_{ij}\, X^L_k\, X^A_j - t_{kj}\, X^A_i\, X^L_j - X^A_k\, X^L_j\, t_{ij} + X^L_i X^A_j\, t_{kj} = 0,$$

which, using $(2.5)_2$, may be written

$$\widehat{B}_{ijkl} - \widehat{B}_{lkji} + t_{ij}\,\delta_{kl} - t_{lk}\,\delta_{ji} - t_{ik}\,\delta_{jl} + t_{ki}\,\delta_{lj} = 0. \tag{5.3}$$

Using (3.6), we see that (5.3) may also be written

$$\widehat{A}_{ijkl} + t_{ij}\,\delta_{kl} = \widehat{A}_{klji} + t_{kl}\,\delta_{ji}. \tag{5.4}$$

We state our results now as two theorems:

Theorem II. *Necessary and sufficient that a given hypo-elastic material which is Cauchy-elastic be also Green-elastic is that* (5.4) *be satisfied for all values of stress that occur.*

Theorem III. *Necessary and sufficient that all hypo-elastic materials corresponding to* (3.1) *or* (3.5) *be elastic in the sense of Green is that* (4.4) *and* (5.4) *be satisfied identically in* t_{ij}.

We see that (3.7) satisfies (5.4) for Cauchy-elastic solutions, *i.e.*, when (4.5) is satisfied. Hence Theorem II implies that hypo-elastic materials corresponding to (3.7) are elastic in the sense of GREEN if ever the stress is a hydrostatic pressure, which is to be expected since they are perfect fluids in that case. If ever, on the other hand, the stress is not a hydrostatic pressure, then the material is not elastic even in the sense of CAUCHY. Thus the stress-configuration class can be very important in determining the nature of the material corresponding to a given set of hypo-elastic equations.

6. An energetic condition for elasticity in the sense of Green

The work done by the stresses* in any material during a motion occuring in the time interval $t_1 \leqq t \leqq t_2$ is given by

$$\int_{t_1}^{t_2}\int_{V} t_{ij}\, d_{ij}\, dV\, dt = \int_{t_1}^{t_2}\int_{V_0} t_{ij}\, d_{ij}\, \frac{\varrho_0}{\varrho}\, dV_0\, dt, \tag{6.1}$$

where V is the volume of material at time t and V_0 is the volume for some fixed reference configuration, for which the density is ϱ_0. In this section we shall prove the following:

Theorem IV. *If for a given hypo-elastic material the work done by the stresses is non-negative for all (kinematically possible) motions for which the initial and final configurations are the same, then this material is elastic in the sense of Green.*

Before proving the theorem, we shall make some observations and establish some lemmas.

Consider a given representation of a hypo-elastic material and a given material point $\underset{0}{X}$. Let $\{{}'t_{ij}, {}'x^k_A\}$ be an element of the corresponding stress-configuration class. Suppose that $[{}''t_{ij}, {}''x^k_A]$ is a stress-displacement gradient pair for which there exist piecewise smooth functions $\hat{\hat{t}}_{ij}(t)$, $\hat{\hat{x}}^k_A(t)$, $t_1 \leqq t \leqq t_2$, satisfying (3.5) and such that

$$[\hat{\hat{t}}_{ij}(t_1), \hat{\hat{x}}^k_A(t_1)] = [{}'t_{ij}, {}'x^k_A]\,(\underset{0}{X}),$$

and

$$[\hat{\hat{t}}_{ij}(t_2), \hat{\hat{x}}^k_A(t_2)] = [{}''t_{ij}, {}''x^k_A].$$

Then we may find an element $\{{}''t_{ij}, {}''x^k_A\}$ of the stress-configuration class corresponding to our given representation which takes the value $[{}''t_{ij}, {}''x^k_A]$ at $\underset{0}{X}$, as we shall now demonstrate.

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* § 27 of [1952].

Let ${}'\tilde{x}^k_A(X)$ be the values of the displacement gradients corresponding to $\{{}'t_{ij}, {}'x^k_A\}$, and let ${}'\underset{0}{X}{}^A_k$ be a set of numbers defined by

$$ {}'\underset{0}{X}{}^A_k \, {}'\tilde{x}^k_B\big(\underset{0}{X}\big) = \delta^A_B, $$

i.e., the inverse displacement gradients at $\underset{0}{X}$ corresponding to $\{{}'t_{ij}, {}'x^k_A\}$. Then define the function

$$ \hat{\tilde{x}}^k_A(X,t) = \hat{\tilde{x}}^k_B(t) \, {}'\underset{0}{X}{}^B_l \, {}'\tilde{x}^l_A(X), \qquad t_1 \leqq t \leqq t_2. \tag{6.2} $$

We see that (6.2) satisfies (3.9), since ${}'\tilde{x}^l_A(X)$ does by assumption; that

$$ \hat{\tilde{x}}^k_A\big(\underset{0}{X}, t\big) = \hat{\tilde{x}}^k_A(t); $$

and that

$$ \hat{\tilde{x}}^k_A(X, t_1) = {}'\tilde{x}^k_A(X) $$

for all X. Hence (6.2) defines a kinematically possible motion in which the values of the displacement gradients are prescribed at $\underset{0}{X}$. And to find the stress-configuration pair $\{{}''t_{ij}, {}''x^k_A\}$ we have but to put (6.2) into (3.5), take $\{{}'t_{ij}, {}'x^k_A\}$ as initial conditions at $t = t_1$, solve (3.5) for $t_1 \leqq t \leqq t_2$, and determine the stress-configuration pair at $t = t_2$.

For a given representation of a hypo-elastic material, we may speak of the allowable stress-displacement gradient pairs at each material point as the possible values that the stress-configuration pairs may take there. And, from the foregoing discussion, these allowable stress-displacement gradient pairs will be precisely all those which may be reached from any one of them along a solution of (3.5).

Moreover, an examination of (6.2) will show that an assignment of $\hat{\tilde{x}}^k_A(t)$ at $\underset{0}{X}$ such that $\hat{\tilde{x}}^k_A(t_1) = \hat{\tilde{x}}^k_A(t_2)$ may be imbedded in a motion of the entire set of material points for which the initial and final configurations coincide. Thus if it were possible to find a solution $\big[\overset{\approx}{t}_{ij}(t), \tilde{\tilde{x}}^k_A(t)\big]$, $t_1 \leqq t \leqq t_2$, of (3.5), where $\big[\overset{\approx}{t}_{ij}(t_1), \tilde{\tilde{x}}^k_A(t_1)\big]$ is an allowable stress-displacement gradient pair at some given material point $\underset{0}{X}$ for some given representation of a hypo-elastic material, such that ${}^{\approx}w < 0$, where

$$ {}^{\approx}w = \int_{t_1}^{t_2} \overset{\approx}{t}_{ij} \, \overset{\approx}{d}_{ij} \, \frac{\varrho_0}{\overset{\approx}{\varrho}} \, dt, \tag{6.3} $$

then for some material volume containing $\underset{0}{X}$ it would, by (6.1) and our foregoing remarks, be possible to find a motion for which, though the initial and final configurations are the same, the work done by the stresses is negative. Hence we may make the following

Remark 1. *Suppose given a representation of a hypo-elastic material for which the hypotheses of Theorem* IV *hold, henceforth called a* representation IV. *Then the quantity* ${}^{\approx}w$ *defined in* (6.3), *and henceforth called the* work density, *must be non-negative for any set of continuous, piecewise smooth functions* $\overset{\approx}{t}_{ij}(t)$, $\tilde{\tilde{x}}^k_A(t)$, $t_1 \leqq t \leqq t_2$, *satisfying* (3.5) *for which* $\big[\overset{\approx}{t}_{ij}(t_1), \tilde{\tilde{x}}^k_A(t_1)\big]$ *is an allowable stress-displacement gradient pair at any given material point* X *such that* $\tilde{\tilde{x}}^k_A(t_1) = \tilde{\tilde{x}}^k_A(t_2)$.

And this shall be our starting point to establish Theorem IV.

Note that (6.3) may be written

$$^{\approx}w = \int_{t_1}^{t_2} \det|\tilde{\tilde{x}}^k_B|\, \tilde{\tilde{t}}_{ij}\, \tilde{\tilde{X}}^A_j \frac{d\tilde{\tilde{x}}^i_A}{dt}\, dt, \tag{6.4}$$

the total derivative replacing the material derivative because, since X is fixed, x^i_A becomes a function of t alone. We now proceed to our lemmas.

Lemma 1. *If* $[\hat{t}_{ij}(t), \hat{x}^k_A(t)]$ *satisfy* (3.5), $t_1 \leqq t \leqq t_2$, *and if* $\overset{*}{t}(t)$ *is a smooth function of* t, $t_1 \leqq t \leqq t_2$, $d\overset{*}{t}/dt \neq 0$ *with, say,* $\overset{*}{t}(t_1) = \underset{*}{t}_1$, $\overset{*}{t}(t_2) = \underset{*}{t}_2$, *then* $\overset{*}{t}_{ij}(\underset{*}{t})$, $\overset{*}{x}^k_A(\underset{*}{t})$ *defined by*

$$\overset{*}{t}_{ij}(\underset{*}{t}) = \hat{t}_{ij}(t), \quad \overset{*}{x}^k_A(\underset{*}{t}) = \hat{x}^k_A(t), \quad \underset{*}{t} = \overset{*}{t}(t),$$

satisfy (3.5) *for* $\min(\underset{*}{t}_1, \underset{*}{t}_2) \leqq \underset{*}{t} \leqq \max(\underset{*}{t}_1, \underset{*}{t}_2)$. *Also the corresponding values of the work density, namely*

$$^{\frown}w \equiv \int_{t_1}^{t_2} \det|\hat{x}^k_B|\, \hat{t}_{ij}\, \hat{X}^A_j \frac{d\hat{x}^i_A}{dt}\, dt,$$

and (6.5)

$$^{*}w \equiv \int_{\underset{*}{t}_1}^{\underset{*}{t}_2} \det|\overset{*}{x}^k_B|\, \overset{*}{t}_{ij}\, \overset{*}{X}^A_j \frac{d\overset{*}{x}^i_A}{d\underset{*}{t}}\, d\underset{*}{t},$$

are related by

$$^{*}w = {}^{\frown}w \qquad if \quad \frac{d\overset{*}{t}}{dt} > 0 \qquad (\text{i.e. } \underset{*}{t}_1 < \underset{*}{t}_2)$$

or

$$^{*}w = -{}^{\frown}w \quad if \quad \frac{d\overset{*}{t}}{dt} < 0 \qquad (\text{i.e. } \underset{*}{t}_2 < \underset{*}{t}_1).$$

Proof. This lemma is readily verified by substitution in (3.5) and (6.5). *q.e.d.*

It follows from Lemma 1 that, without loss of generality, we may assume that $t_1 = 0$ and $t_2 = 1$ where convenient. We proceed to

Lemma 2. *Consider a representation* IV *and any material point* $\underset{0}{X}$. *Let* $[\hat{t}_{ij}(t), \hat{x}^k_A(t)]$, $t_1 \leqq t \leqq t_2$, *be any solution of* (3.5) *with* $[\hat{t}_{ij}(t_1), \hat{x}^k_A(t_1)]$ *an allowable stress-displacement gradient pair, and* $\hat{x}^k_A(t_1) = \hat{x}^k_A(t_2)$. *Then* $^{\frown}w = 0$, *where* $^{\frown}w$ *is the corresponding value of the work density.*

Proof. Using Lemma 1 we may obtain a second solution of (3.5), $[\overset{*}{t}_{ij}(\underset{*}{t}), \overset{*}{x}^k_A(\underset{*}{t})]$, $t_1 \leqq \underset{*}{t} \leqq t_2$, obeying the hypotheses of Lemma 2, by writing

$$\overset{*}{t}(t) \equiv t_1 + t_2 - t.$$

Let $^{*}w$ be the value of the work density corresponding to this second solution. Then by Lemma 1 $w = -{}^{*}w$. But, because of Remark 1, $w \geqq 0$ and $^{*}w \geqq 0$. Hence $w = {}^{*}w = 0$. *q.e.d.*

Lemma 3. *Consider a representation* IV. *Let* $[\underset{1}{t}_{ij}, \underset{1}{x}^k_A]$ *and* $[\underset{2}{t}_{ij}, \underset{2}{x}^k_A]$ *be two allowable stress-displacement gradient pairs at a given material point. Suppose* *that* $[\hat{t}_{ij}(t), \hat{x}^k_A(t)]$ *and* $[\hat{\hat{t}}_{ij}(\tau), \hat{\hat{x}}^k_A(\tau)]$ *are two solutions of* (3.5), $t_1 \leqq t \leqq t_2$, $\tau_1 \leqq \tau \leqq \tau_2$, *such that* $[\hat{t}_{ij}(t_1), \hat{x}^k_A(t_1)] = [\underset{1}{t}_{ij}, \underset{1}{x}^k_A] = [\hat{\hat{t}}_{ij}(\tau_1), \hat{\hat{x}}^k_A(\tau_1)]$, $\hat{x}^k_A(t_2) = \underset{2}{x}^k_A = \hat{\hat{x}}^k_A(\tau_2)$. *Let the corresponding values of the work density be* ${}^{\frown}w$ *and* ${}^{\hat{\frown}}w$ *respectively. Then* ${}^{\frown}w = {}^{\hat{\frown}}w$.

Proof. By Lemma 1 we may, without loss of generality, take $t_1 = \tau_1 = 0$ and $t_2 = \tau_2 = 1$. Then we define another solution of (3.5), $[\breve{t}_{ij}({}'t), \breve{x}^k_A({}'t)]$, $0 \leqq {}'t \leqq 2$, as follows:

$$\breve{t}_{ij}({}'t) = \hat{t}_{ij}(1 - {}'t), \qquad \breve{x}^k_A({}'t) = \hat{x}^k_A(1 - {}'t), \qquad 0 \leqq {}'t \leqq 1,$$
$$\breve{t}_{ij}({}'t) = \hat{\hat{t}}_{ij}({}'t - 1), \qquad \breve{x}^k_A({}'t) = \hat{\hat{x}}^k_A({}'t - 1), \qquad 1 \leqq {}'t \leqq 2.$$

That this is a solution in each of the intervals $0 \leqq {}'t \leqq 1$ and $1 \leqq {}'t \leqq 2$ may be verified immediately with the use of Lemma 1. Also it is continuous at ${}'t = 1$ because, by the hypotheses of our lemma and the definition of this solution, $[\hat{t}_{ij}(0), \hat{x}^k_A(0)] = [\hat{\hat{t}}_{ij}(0), \hat{\hat{x}}^k_A(0)] = [\breve{t}_{ij}(1), \breve{x}^k_A(1)]$. Hence, since $\breve{x}^k_A(0) = \hat{x}^k_A(1) = \underset{1}{x}^k_A = \hat{\hat{x}}^k_A(1) = \breve{x}^k_A(2)$, Lemma 2 yields

$$\breve{}w \equiv \int_0^2 \det|\breve{x}^k_B|\, \breve{t}_{ij}\, \breve{X}^A_j \frac{d\breve{x}^i_A}{d{}'t}\, d{}'t = 0. \tag{6.6}$$

But from Lemma 1

$$\begin{aligned} \int_0^1 \det|\breve{x}^k_B|\, \breve{t}_{ij}\, \breve{X}^A_j \frac{d\breve{x}^i_A}{d{}'t}\, d{}'t &= -{}^{\frown}w, \\ \int_1^2 \det|\breve{x}^k_B|\, \breve{t}_{ij}\, \breve{X}^A_j \frac{d\breve{x}^i_A}{d{}'t}\, d{}'t &= {}^{\hat{\frown}}w. \end{aligned} \tag{6.7}$$

Combining (6.6) and (6.7) gives

$$0 = \breve{}w = {}^{\hat{\frown}}w - {}^{\frown}w,$$

and hence the lemma. *q.e.d.*

Remark 2. *Suppose that we consider a given stress-configuration pair,* $\{\underset{0}{t}_{ij}, \underset{0}{x}^k_A\}$, *corresponding to a given representation* IV. *Then, by Lemma* 3, *we may define a function* $\tilde{\Sigma}(x^i_A, X)$ *as follows: for each fixed* X, $\tilde{\Sigma}({}'x^i_A, X)$ *is the value of the work density for any solution of* (3.5) *for which* $[\underset{0}{t}_{ij}, \underset{0}{x}^k_A](X)$ *constitute the initial conditions and for which the final values of the displacement gradients are* ${}'x^i_A$.

We cannot differentiate $\tilde{\Sigma}$ with respect to x^i_A by ordinary means since $\tilde{\Sigma}$ is defined by an integral whose integrand has not been shown to be a function of x^i_A and X. Nevertheless, we shall now prove

Lemma 4. *The partial derivatives of* $\tilde{\Sigma}(x^i_A, X)$ *with respect to the displacement gradients exist and are given by*

$$\frac{\partial \tilde{\Sigma}}{\partial x^i_A} = \det|x^k_B|\, t_{ij} X^A_j, \tag{6.8}$$

where, for given x^k_A, t_{ij} *has any value such that* $[t_{ij}, x^k_A]$ *is an allowable stress-displacement gradient pair at* X.

Proof. Let $[\underset{1}{t}_{ij}, \underset{1}{x}^k_A]$ be any allowable stress-displacement gradient pair at X. Then there exists a solution of (3.5) $[\hat{t}_{ij}(t), \hat{x}^k_A(t)]$, $0 \leqq t \leqq 1$, such that

$$[\hat{t}_{ij}(0), \hat{x}^k_A(0)] = [\underset{0}{t}_{ij}, \underset{0}{x}^k_A](X),$$
$$[\hat{t}_{ij}(1), \hat{x}^k_A(1)] = [\underset{1}{t}_{ij}, \underset{1}{x}^k_A].$$

By definition

$$\tilde{\Sigma}(x^k_A, X) = \int_0^1 \det|\hat{x}^k_B|\, \hat{t}_{ij}\, \hat{X}^A_j \frac{d\hat{x}^i_A}{dt}\, dt.$$

Let us extend the function $\hat{x}^i_A(t)$ beyond $t=1$ in any one of the following eighteen ways; write

$$\hat{x}^i_A(t) = \underset{1}{\hat{x}}^i_A + \varepsilon(t-1)\,\delta^i_p\,\delta^L_A, \qquad 1 \leqq t \leqq 1+h, \tag{6.9}$$

where p and L each take *one* of the values 1, 2, 3, ε takes either the values $+1$ or -1, and h is some positive real number. Let $t_{ij}(t)$ be the corresponding solution of (3.5). Then

$$\begin{aligned}\tilde{\Sigma}(\underset{1}{x}^i_A + \varepsilon\,\eta\,\delta^i_p\,\delta^L_A, X) - \tilde{\Sigma}(\underset{1}{x}^i_A, X) &= \int_1^\eta \det|\hat{x}^k_B(t)|\, \hat{t}_{pj}(t)\, \hat{X}^L_j(t)\,\varepsilon\, dt \\ &= \det|\hat{x}^k_B(\tau)|\, \hat{t}_{pj}(\tau)\, \hat{X}^L_j(\tau)\,\varepsilon\,(\eta-1),\end{aligned} \tag{6.10}$$

where $1 < \eta \leqq 1+h$, and τ is some value between 1 and η, the last inequality following from the law of the mean for integrals. Using (6.9), (6.10) becomes

$$\frac{\tilde{\Sigma}(\hat{x}^i_A(\eta), X) - \tilde{\Sigma}(\underset{1}{\hat{x}}^i_A, X)}{x^p_L(\eta) - \underset{1}{x}^p_L} = \det|\hat{x}^k_B(\tau)|\, \hat{t}_{pj}(\tau)\, \hat{X}^L_j(\tau), \qquad 1 < \tau < \eta. \tag{6.11}$$

Since the limit of the right-hand side of (6.10) exists as $\eta \to 1$, the limit of the left-hand side must also exist. Since $\hat{x}^i_A(\eta) = \underset{1}{x}^i_A$ for $\eta \geqq 1$ unless $i=p$ and $A=L$, the limit of the left-hand side of (6.10) as $\eta \to 1$ represents the right or left-sided partial derivative of $\tilde{\Sigma}$ with respect to x^p_L according to whether ε is $+1$ or -1. But both of these limits are equal and, in fact, are given by the right-hand side of (6.11) evaluated at $\tau = 1$, since solutions to (3.5) must be continuous in t. Hence the partial derivatives of $\tilde{\Sigma}$ with respect to the displacement gradients exist and are given by (6.8), where t_{ij} is determined as in the hypotheses of Lemma 4. *q.e.d.*

With the help of our lemmas, Theorem IV follows immediately. From Lemma 4, in particular (6.8), and from $(2.5)_2$, a stress which will make $[t_{ij}, x^k_A]$ an allowable stress-displacement gradient pair at any given X, for a given representation IV, must satisfy

$$t_{ij} = [\det|x^l_L|]^{-1}\, x^j_A \frac{\partial\tilde{\Sigma}(x^k_B, X)}{\partial x^i_A} = \frac{\varrho}{\varrho_0}\, x^j_A \frac{\partial\tilde{\Sigma}(x^k_B, X)}{\partial x^i_A}, \tag{6.12}$$

and thus there can be but one such stress. Moreover, comparison of (6.11) with (2.13) shows that a material obeying the hypotheses of Theorem IV must be elastic in the sense of GREEN. Hence Theorem IV is established.

7. Isotropy

Consider a representation of a Cauchy-elastic material in which the material coordinates are one possible set of values of the spatial coordinates and are cartesian. Let O^A_B and O^A, $A, B = 1, 2, 3$, be a set of constants such that

$$\delta_{AB}\, O^A_C\, O^B_D = \delta_{CD}.$$

Then

$$'X^A = O^A_B\, X^B + O^A \tag{7.1}$$

denotes an orthogonal transformation of material points so that

$$'x^i_A \equiv \frac{\partial x^i}{\partial\, 'X^A} = x^i_B\, O^B_A, \qquad 'X^A_i \equiv \frac{\partial\, 'X^A}{\partial x^i} = X^B_i\, O^A_B. \tag{7.2}$$

If in (2.8) we have

$$\hat{t}_{ij}('x^i_A, 'X^B) = \hat{t}_{ij}(x^i_A, X^B), \tag{7.3}$$

when X^A and $'X^A$ are related by (7.1), we say that the given Cauchy-elastic material is isotropic with respect to the given representation.

It is well known [1952] [1955, 3] that the above definition of isotropy requires that the stress t_{ij} be a uniform hydrostatic pressure when $x^i_A = \delta^i_A$. But the converse of this statement is not true, as can be verified by considering a Green-elastic material for which

$$\tilde{\Sigma} \equiv [(\delta_{kl}\, x^k_A\, x^l_B - \delta_{AB})\, H^{AB}]^2, \tag{7.4}$$

where $H^{11} = 1$, $H^{22} = 2$, $H^{33} = 1$ and $H^{AB} = 0$ if $A \neq B$. That (7.4) is consistent with the principle of isotropy of space may be checked using the last paragraph in § 2.

Using (2.13), the stress for (7.4) becomes

$$t^{ij} = 2\,\frac{\varrho}{\varrho_0}\,(\delta_{kl}\, x^k_A\, x^l_B - \delta_{AB})\, H^{AB}\, x^i_L\, H^{LM}\, x^j_M. \tag{7.5}$$

Consider the following two special cases of (2.1);

$$x^1 = X^1, \qquad x^2 = 2X^2, \qquad x^3 = X^3 \qquad \text{(case 1)}$$

and

$$x^1 = {'X^2}, \qquad x^2 = 2\,{'X^1}, \qquad x^3 = {'X^3} \qquad \text{(case 2)}$$

which differ by the orthogonal transformation

$$'X^1 = X^2, \qquad 'X^2 = X^1, \qquad 'X^3 = X^3.$$

We obtain from (7.5)

$$t^{11} = 6, \qquad t^{22} = 8, \qquad t^{33} = 6, \qquad t^{ij} = 0, \qquad i \neq j \qquad \text{(case 1)}$$

$$t^{11} = 12, \qquad t^{22} = 6, \qquad t^{33} = 3, \qquad t^{ij} = 0, \qquad i \neq j \qquad \text{(case 2)}$$

so that the Green-elastic material given by (7.4) is not isotropic with respect to the given representation although from (7.5) it is readily seen that the stress vanishes, and hence is certainly a hydrostatic pressure, when $x^i_A = \delta^i_A$.

We shall now establish the following:

Theorem V. *Given a representation of a Cauchy-elastic material which is also hypo-elastic, this material is also isotropic with respect to this representation if and only if the stress is a uniform hydrostatic pressure when* $x^i_A = \delta^i_A$.

Proof. Necessity has been established elsewhere, as stated above. We have to prove sufficiency. Assume, then, that the stress is a uniform hydrostatic pressure when $x^i_A = \delta^i_A$.

Let (2.8) correspond to our representation and define

$$\hat{\hat{t}}_{ij}(x^k_A, X^C) \equiv \hat{t}_{ij}('x^k_A, 'X^C), \tag{7.6}$$

using (7.1) and (7.2). Then, by (7.2),

$$\hat{\hat{t}}_{ij}(x^i_A, X^C) = \hat{t}_{ij}(O^H_A\, x^i_H, O^C_H\, X^H + O^C), \tag{7.7}$$

and by substitution into (4.2) we find that $\hat{\hat{t}}_{ij}$ is a solution. But when $x^i_A = \delta^i_A$ we have $t_{ij} = -p\,\delta_{ij}$, where p is independent of X. Hence, using (2.11) and (2.12), we see that

$$-p\,\delta^{ij} = \hat{t}^{ij}(\delta^k_A, X^C) = \delta^i_A\,\delta^j_B\,\hat{T}^{AB}(\delta^K_p\,\delta^L_q\,\delta^{pq}, X^C) = \delta^i_A\,\delta^j_B\,\hat{T}^{AB}(\delta^{KL}, X^C), \tag{7.8}$$

from which

$$\hat{T}^{AB}(\delta^{KL}, X^C) = -p\,\delta^{AB} = T^{AB}(\delta^{KL}, 'X^C), \tag{7.9}$$

the last equality following since p is independent of X. Thus, putting $x^i_A = \delta^i_A$ into (7.6) and using (7.7), (7.8) and (7.9), we obtain

$$\begin{aligned}\hat{\hat{t}}^{ij}(\delta^k_A, X^C) &= \hat{t}^{ij}(\delta^k_B\,O^B_A, 'X^C) = \delta^i_R\,O^R_A\,\delta^j_S\,O^S_B\,\hat{T}^{AB}(\delta^M_p\,O^K_M\,\delta^N_s\,O^L_N\,\delta^{ps}, 'X^C)\\ &= \delta^i_R\,O^R_A\,\delta^j_S\,O^S_B\,\hat{T}^{AB}(\delta^{KL}, 'X^C) = -p\,\delta^i_R\,O^R_A\,\delta^j_S\,O^S_B\,\delta^{AB}\\ &= -p\,\delta^i_R\,\delta^j_S\,\delta^{RS} = -p\,\delta^{ij} = \hat{t}^{ij}(\delta^k_A, X^C).\end{aligned}$$

Since the two solutions of (4.2), $\hat{t}^{ij}$ and $\hat{\hat{t}}^{ij}$, coincide for one value of x^i_A at each X, they must, because of uniqueness of solutions to (4.2), coincide for all values of x^i_A at each X, *i.e.*,

$$\hat{t}^{ij}('x^k_A, 'X^C) = \hat{\hat{t}}^{ij}(x^k_A, X^C) = \hat{t}^{ij}(x^k_A, X^C),$$

and hence the theorem is established.

We remark, then, that the material (7.4) is elastic but not hypo-elastic, since if it were hypo-elastic it would have to be isotropic because of Theorem V, and it is not. Thus hypo-elasticity does not include all of elasticity.

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Inversion of Lambda-Matrices and Application to the Theory of Linear Vibrations

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1. Introduction

We consider the problem of inverting the matrix

$$D^{(l)}(\lambda) \equiv (A_0 \lambda^l + A_1 \lambda^{l-1} + \cdots + A_{l-1} \lambda + A_l), \tag{1}$$

where $A_0, A_1, \ldots, A_l$ are $(n \times n)$ matrices of real elements, and λ is a scalar. The special case $l=1$ is comparatively well known, for (assuming A_0 to be non-singular) it can be reduced to the problem of finding the inverse matrix of $(I\lambda - A)$, known as the resolvent of A (see [*1*]). The resolvent of A can be viewed as a series of partial fractions with denominators $(\lambda - \lambda_i)^j$, where λ_i are the latent roots of A. In the same way the inverse of matrix (1) may be expanded as a series of partial fractions involving its latent roots. This was done by Frazer, Duncan & Collar [*2*]. We are to develop a rather different form for the inverse of (1) which has the advantage of being developed entirely in matrix form, and which is unaffected by the presence of a latent root of multiplicity α (>1) provided (1) has degeneracy α when evaluated at this latent root.

The method consists in first finding the inverse of $D^{(1)}(\lambda)$, in which case the matrix elements are linear in λ; and then for $l > 1$ we transform the problem to that of inverting a matrix of order ln whose elements are again linear in λ. The application to the theory of linear vibrations arises when $l=2$.

We shall assume throughout this paper that the leading matrix, A_0, is non-singular. Consider the set of homogeneous equations

$$(A_0 \lambda^l + A_1 \lambda^{l-1} + \cdots + A_l)\, \boldsymbol{q} = 0, \tag{2}$$

and the set obtained from the transpose of (1),

$$(A_0' \lambda^l + A_1' \lambda^{l-1} + \cdots + A_l')\, \boldsymbol{r} = 0, \tag{3}$$

where $\boldsymbol{q}$ and $\boldsymbol{r}$ are $(n \times 1)$ column vectors. The latent roots of matrix (1) are the values of λ for which its determinant is zero and for which (2) and (3) have non-trivial solutions, known as the latent vectors. If we count each root according to its multiplicity, there will be ln latent roots, λ_i, in all, each with associated latent vectors $\boldsymbol{q}_i$ and $\boldsymbol{r}_i$. We shall always assume that if a root of multiplicity α occurs $(1 \leqq \alpha \leqq n)$, λ_m say, then $D^{(l)}(\lambda_m)$ has degeneracy α. That is, there are

α *linearly independent* solutions of equations (2) and (3) when λ is replaced by λ_m*. We arrange the latent roots and vectors in some fixed order. Define the $(n \times ln)$ modal matrices, Q and R, as the matrices whose columns are the latent vectors of (2) and (3) respectively. We also define the $(ln \times ln)$ *diagonal* matrix Λ, so that its i^{th} diagonal element is λ_i, the i^{th} latent root, and note that the elements of Q, R, and Λ may be complex numbers.

2. Inversion of lambda-matrices

The following theorem is not new in principle, but the formulation and proof given below are the most appropriate for the present work.

Theorem 1. *If A and C are real $(n \times n)$ matrices, of which A is non-singular, then the latent vectors of the equations*

$$(A\lambda + C)\,\boldsymbol{q} = 0, \quad \textit{and} \quad (A'\lambda + C')\,\boldsymbol{r} = 0 \tag{4}$$

can be normalised in such a way that

$$R'AQ = I, \quad \textit{and} \quad R'CQ = -\Lambda, \tag{5}$$

in which case

$$(A\lambda + C)^{-1} = Q(I\lambda - \Lambda)^{-1}R', \qquad \textit{if } \lambda \neq \lambda_i, \tag{6}$$

and provided that for any latent root λ_m of multiplicity α $(1 \leqq \alpha \leqq n)$, the matrix $(A\lambda_m + C)$ has degeneracy α.

Proof. We notice that in this case Q, R, and Λ are all $(n \times n)$ square matrices. It follows from our definitions that

$$AQ\Lambda + CQ = 0,$$

and

$$A'R\Lambda + C'R = 0.$$

Premultiply these equations by R' and Q' respectively, and after transposing the second we obtain

$$(R'AQ)\Lambda + R'CQ = 0 \quad \text{and} \quad \Lambda(R'AQ) + R'CQ = 0. \tag{7}$$

Thus,

$$(R'AQ)\Lambda = \Lambda(R'AQ). \tag{8}$$

If no two of the latent roots are equal, this can only be true if $(R'AQ)$ is a diagonal matrix. If there is a single root of multiplicity α $(1 \leqq \alpha \leqq n)$ among the latent roots, we order the n latent roots so that the α equal roots take the first α positions. From (8) we then deduce that the leading $(\alpha \times \alpha)$ minor of $(R'AQ)$ is the only part of that matrix which is not in diagonal form. However, for arbitrary constants $\beta_1, \beta_2, \ldots, \beta_\alpha$ and $\gamma_1, \gamma_2, \ldots, \gamma_\alpha$, we have

$$(A\lambda_1 + C)(\beta_1 \boldsymbol{q}_1 + \beta_2 \boldsymbol{q}_2 + \cdots + \beta_\alpha \boldsymbol{q}_\alpha) = 0,$$

and

$$(A'\lambda_1 + C')(\gamma_1 \boldsymbol{q}_1 + \gamma_2 \boldsymbol{q}_2 + \cdots + \gamma_\alpha \boldsymbol{r}_\alpha) = 0.$$

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* The problem which stimulated this work was that of linear mechanical vibrations. The two strong conditions I have made — that A_0 is non-singular, and that $D^{(l)}(\lambda_m)$ is of degeneracy α — will invariably be satisfied in these applications. The inherent simplicity of this important case seems to be lost in the treatment of FRAZER, DUNCAN & COLLAR.

Thus any linear combination of the first α latent vectors is also a latent vector with latent root λ_1. By hypothesis the matrix $(A\lambda_1 + C)$ has degeneracy α, and this implies that the vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_\alpha$ are linearly independent, and so are $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_\alpha$. We may therefore use the indeterminacy of these vectors to complete the biorthogonal system which will reduce $(R' A Q)$ to diagonal form. The extension to cases with more than one multiple root follows immediately. We then obtain the first of equations (5) by an appropriate normalisation of the latent vectors. The second of equations (5) then follows from (7).

Using equations (5) we now see that

$$R'(A\lambda + C)\,Q = I\lambda - \Lambda.$$

If $\lambda \neq \lambda_i$, we may invert both sides of this equation and obtain

$$Q^{-1}(A\lambda + C)^{-1}(R')^{-1} = (I\lambda - \Lambda)^{-1},$$

which implies equation (6) and completes the proof.

Theorem 2. *If A_0 is non-singular and $D^{(l)}(\lambda_i)$ has degeneracy equal to the multiplicity of λ_i for $i = 1, 2, \ldots, ln$, then the latent vectors of equations* (2) *and* (3) *can be normalised in such a way that*

$$\lambda^r [D^{(l)}(\lambda)]^{-1} = Q\Lambda^r (I\lambda - \Lambda)^{-1} R', \qquad \text{for } r = 0, 1, 2, \ldots, (l-1), \tag{9}$$

and

$$\lambda^l [D^{(l)}(\lambda)]^{-1} = Q\Lambda^l (I\lambda - \Lambda)^{-1} R' + A_0^{-1} \tag{10}$$

provided $\lambda \neq \lambda_i$.

Proof. (We assume from this point onwards that $l \geqq 2$.) We note that Q and R are now $(n \times ln)$ matrices, and Λ is an $(ln \times ln)$ matrix. For any latent vector $\boldsymbol{x}$ (of either (2) or (3)) we denote by $\boldsymbol{x}^{(s)}$ the product $\lambda^s \boldsymbol{x}$, λ being the corresponding latent root. It can easily be verified that (2) is equivalent to the following partitioned matrix equation (suggested by Foss [3]) whose elements are linear functions of λ.

$$\left\{ \begin{aligned} &\begin{bmatrix} 0 & 0 & \cdots & & 0 & A_0 \\ 0 & 0 & \cdots & & A_0 & A_1 \\ \vdots & \vdots & & A_0 & A_1 & A_2 \\ & & & & \vdots & \vdots \\ 0 & 0 & A_0 & & \cdot & \cdot \\ 0 & A_0 & A_1 & \cdots & & \cdot \\ A_0 & A_1 & A_2 & \cdots & \cdot & A_{l-1} \end{bmatrix} \lambda + \\ &+ \begin{bmatrix} 0 & 0 & \cdots & & 0 & -A_0 & 0 \\ 0 & 0 & \cdots & & -A_0 & -A_1 & 0 \\ \vdots & \vdots & & -A_0 & -A_1 & -A_2 & 0 \\ & & & & & \vdots & \vdots \\ & -A_0 & & & & -A_{l-3} & 0 \\ -A_0 & -A_1 & \cdot\cdot & & -A_{l-3} & -A_{l-2} & 0 \\ 0 & 0 & \cdot\cdot & & 0 & 0 & A_l \end{bmatrix} \end{aligned} \right\} \begin{bmatrix} \boldsymbol{q}^{(l-1)} \\ \boldsymbol{q}^{(l-2)} \\ \vdots \\ \boldsymbol{q}^{(1)} \\ \boldsymbol{q} \end{bmatrix} = 0. \tag{11}$$

Define the two partitioned matrices appearing in this equation as $\mathscr{A}$ and $\mathscr{C}$ respectively. Since A_0 is non-singular, we see immediately that $\mathscr{A}$ is non-singular, and that $\mathscr{C}$ is non-singular if A_0 *and* A_l are non-singular. Further, if all the matrices A_i are symmetric, then so are $\mathscr{A}$ and $\mathscr{C}$. Define the $(ln \times ln)$ modal matrices associated with $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{A}'$, $\mathscr{C}'$ to be $\mathscr{Q}$ and $\mathscr{R}$ respectively; where, from our definitions

$$\mathscr{Q} = \begin{bmatrix} Q\Lambda^{l-1} \\ \vdots \\ Q\Lambda \\ Q \end{bmatrix} \quad \text{and} \quad \mathscr{R} = \begin{bmatrix} R\Lambda^{l-1} \\ \vdots \\ R\Lambda \\ R \end{bmatrix}. \tag{12}$$

If λ_m is a latent root of (2) with multiplicity α $(1 < \alpha \leqq n)$, then by hypothesis there are α linearly independent latent vectors $\boldsymbol{q}_i$ associated with λ_m. It is clear that the corresponding columns of $\mathscr{Q}$ are linearly independent also, so that we have α independent latent vectors of (9) corresponding to the latent root λ_m. That is, $(\mathscr{A}\lambda_m + \mathscr{C})$ has degeneracy α.

The conditions of Theorem 1 are therefore satisfied, and $\mathscr{R}$ can be defined in such a way* that

$$(\mathscr{A}\lambda + \mathscr{C})^{-1} = \mathscr{Q}(I\lambda - \Lambda)^{-1}\mathscr{R}'. \tag{13}$$

For convenience, we define $(\mathscr{A}\lambda + \mathscr{C})^{-1} = \mathscr{B}$ and denote the $(n \times n)$ partitions of $\mathscr{B}$ by B_{ij}. On substituting from (12) into (13)** we find that

$$\begin{aligned} B_{ij} &= Q\Lambda^{l-i}(I\lambda - \Lambda)^{-1}\Lambda^{l-j}R' \\ &= Q\Lambda^{2l-(i+j)}(I\lambda - \Lambda)^{-1}R'. \end{aligned} \tag{14}$$

Thus B_{ij} depends only on the sum $i+j$. That is, there are only $(2l-1)$ distinct partitions of $\mathscr{B}$, given by $(i+j) = 2, 3, \ldots, 2l$.

Now by our definition of $\mathscr{B}$ we have

$$(\mathscr{A}\lambda + \mathscr{C})\,\mathscr{B} = I_{ln},$$

where we use I_{ln} for the unit matrix of order ln, reserving I for the unit matrix of order n. Writing the left-hand side of this relation in partitioned form we get

$$\left[\begin{array}{ccccccc} 0 & 0 & \cdot & & 0 & -A_0 & A_0\lambda \\ 0 & 0 & \cdot & & -A_0 & (A_0\lambda - A_1) & A_1\lambda \\ 0 & \cdot & & & (A_0\lambda - A_1) & (A_1\lambda - A_2) & A_2\lambda \\ 0 & & & & \cdot & \cdot & \cdot \\ 0 & -A_0 & (A_0\lambda - A_1) & \cdot & & \cdot & \cdot \\ -A_0 & (A_0\lambda - A_1) & (A_1\lambda - A_2) & \cdot & \cdot & (A_{l-3}\lambda - A_{l-2}) & A_{l-2}\lambda \\ A_0\lambda & A_1\lambda & A_2\lambda & \cdot & \cdot & A_{l-2}\lambda & (A_{l-1}\lambda + A_l) \end{array}\right] \times$$

$$\times [B_{ij}] = I_{ln}. \tag{15}$$

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* The choice involved here concerns the definition of columns of $\mathscr{Q}$ and $\mathscr{R}$ corresponding to a multiple root. If we take *any* linear transformation of these columns, the form of equations (12) will not be destroyed.

** In this and subsequent manipulations of partitioned matrices the matrix partitions can be treated in much the same way as scalar elements. See *e.g.* FRAZER, DUNCAN & COLLAR, *loc. cit.*, § 1.7.

We use this equation to evaluate the partitions B_{il}. The partition of the product on the left appearing in the $(1, l)$ position gives

$$-A_0 B_{l-1,l} + \lambda A_0 B_{l,l} = 0,$$

and since A_0 is non-singular,

$$B_{l-1,l} = \lambda B_{l,l}. \tag{16}$$

The partition of the product in the $(2, l)$ position gives

$$-A_0 B_{l-2,l} + (A_0 \lambda - A_1) B_{l-1,l} + A_1 \lambda B_{l,l} = 0,$$

and using (16), we find that

$$B_{l-2,l} = \lambda^2 B_{l,l}.$$

Continuing in this way up to the partition in the $(l-1, l)$ position, we find that

$$B_{i,l} = \lambda^{l-i} B_{l,l}, \qquad \text{for } i = 1, 2, \ldots, (l-1). \tag{17}$$

From the partition in the (l, l) position we obtain finally

$$A_0 B_{1l} \lambda + A_1 B_{2l} \lambda + \cdots + A_{l-2} B_{l-1,l} \lambda + (A_{l-1} \lambda + A_l) B_{l,l} = I,$$

and using (17),

$$(A_0 \lambda^l + A_1 \lambda^{l-1} + \cdots + A_{l-1} \lambda + A_l) B_{l,l} = I. \tag{18}$$

From (18) and (14) we get

$$B_{l,l} = [D^{(l)}(\lambda)]^{-1} = Q (I\lambda - \Lambda)^{-1} R';$$

and from (17) and (14),

$$B_{i,l} = \lambda^{l-i} [D^{(l)}(\lambda)]^{-1} = Q \Lambda^{l-i} (I\lambda - \Lambda)^{-1} R'$$

for $i = 1, 2, \ldots, (l-1)$. These two results establish equation (9).

To prove equation (10) we consider the partition of the left-hand side of (15) in the $(1, 1)$ position. We obtain

$$-A_0 B_{l-1,1} + A_0 \lambda B_{l,1} = I,$$

or,

$$\lambda B_{l,1} = B_{l-1,1} + A_0^{-1}.$$

But $B_{l,1} = B_{1,l} = \lambda^{l-1} B_{l,l}$, using (14) and (17); so that substituting for $B_{l-1,1}$ from (14), we obtain

$$\lambda^l B_{l,l} = Q \Lambda^l (I\lambda - \Lambda)^{-1} R' + A_0^{-1},$$

and (10) follows from equation (18)*.

Equation (9) with $r = l-1$ (or even equation (10)) may be valuable in obtaining $[D^{(l)}(\lambda)]^{-1}$ for large values of λ. We shall see below that some interesting results are obtained from these equations when λ is zero.

3. Properties of the latent vectors

Applying Theorem 1 to the matrices $\mathscr{A}$, $\mathscr{C}$ again, equation (5) becomes

$$\mathscr{R}' \mathscr{A} \mathscr{Q} = I_{ln}, \quad \text{and} \quad \mathscr{R}' \mathscr{C} \mathscr{Q} = -\Lambda. \tag{19}$$

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* These results can be continued to values of i, j for which $i+j<l$, but they rapidly become more cumbersome and are of doubtful value, so they are not included here.

These are the orthogonality conditions under which (13) is true. We see from equation (7) that if one of these is satisfied then so is the other. We define $(ln \times ln)$ matrices A_i^* by

$$A_i^* = R' A_i Q.$$

Now substitute for $\mathscr{D}$, $\mathscr{R}$, from (12) and for $\mathscr{A}$, $\mathscr{C}$ from (11), and it is found that we can rewrite conditions (19) in the form

$$\begin{aligned} &A_{l-1}^* + (\Lambda A_{l-2}^* + A_{l-2}^* \Lambda) + (\Lambda^2 A_{l-3}^* + \Lambda A_{l-3}^* \Lambda + A_{l-3}^* \Lambda^2) + \cdots \\ &\qquad \cdots + (\Lambda^{l-1} A_0^* + \Lambda^{l-2} A_0^* \Lambda + \cdots + \Lambda A_0^* \Lambda^{l-2} + A_0^* \Lambda^{l-1}) = I_{ln}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &A_l^* - (\Lambda A_{l-2}^* \Lambda) - (\Lambda^2 A_{l-3}^* \Lambda + \Lambda A_{l-3}^* \Lambda^2) - \cdots \\ &\qquad \cdots - (\Lambda^{l-1} A_0^* \Lambda + \Lambda^{l-2} A_0^* \Lambda^2 + \cdots + \Lambda A_0^* \Lambda^{l-1}) = -\Lambda, \end{aligned} \tag{21}$$

and, as noted above, either one of these implies the other. The complication of this normalising condition detracts from the apparent simplicity of equations (9) and (10).

If we pick out the i^{th} diagonal element from (20), we see that

$$\boldsymbol{r}_i' \{A_{l-1} + 2\lambda_i A_{l-2} + \cdots + l \lambda_i^{l-1} A_0\} \boldsymbol{q}_i = 1.$$

Thus the normalisation condition can be written in the scalar form;

$$\boldsymbol{r}_i' \frac{d}{d\lambda} [D^{(l)}(\lambda)]_{\lambda=\lambda_i} \boldsymbol{q}_i = 1, \qquad \text{for } i = 1, 2, \ldots, ln.$$

Of the sets of ln vectors obtained from (2) and (3) at most n can be linearly independent. Relations between these vectors are provided by the $(2l-1)$ matrix conditions obtained by equating the expressions given in (14) to the corresponding expressions obtained by analysis of (15). The statement of Theorem 2 contains the most simple $(l+1)$ relations obtained in this way. However, to be of any value (from this point of view) these conditions should be independent of λ. We obtain $(l+1)$ such conditions by considering the limits of equations (9) and (10) obtained when $\lambda \to 0$. This may lead to difficulties if there are any null latent roots, for these will give rise to at least one element $1/\lambda$ appearing in the matrix $(I\lambda - \Lambda)^{-1}$.

In spite of this we can prove the following:

Theorem 3. *Under the conditions of Theorem 2*

$$Q \Lambda^{l-1} R' = A_0^{-1}, \tag{22}$$

$$Q \Lambda^u R' = 0, \qquad \textit{for } u = 0, 1, 2, \ldots, (l-2); \tag{23}$$

and if in addition A_l *is non-singular, then*

$$Q \Lambda^{-1} R' = -A_l^{-1}. \tag{24}$$

Proof. Putting $r = 0$ in equation (9), we have

$$[D^{(l)}(\lambda)]^{-1} = Q(I\lambda - \Lambda)^{-1} R'. \tag{25}$$

If we now substitute this back into (9) and (10) (with $r \neq 0$), we get

$$\lambda^r Q (I\lambda - \Lambda)^{-1} R' = Q \Lambda^r (I\lambda - \Lambda)^{-1} R' \qquad \text{for } r = 1, 2, \ldots, (l-1),$$

and

$$\lambda^l Q (I\lambda - \Lambda)^{-1} R' = Q \Lambda^l (I\lambda - \Lambda)^{-1} R' + A_0^{-1}.$$

Rearranging these equations, we get

$$Q (I\lambda^r - \Lambda^r)(I\lambda - \Lambda)^{-1} R' = 0, \qquad \text{for } r = 1, 2, \ldots, (l-1),$$

and

$$Q (I\lambda^l - \Lambda^l)(I\lambda - \Lambda)^{-1} R' = A_0^{-1}.$$

Thus,

$$Q (I\lambda^{r-1} + \lambda^{r-2}\Lambda + \cdots + \Lambda^{r-1}) R' = 0, \qquad \text{for } r = 1, 2, \ldots, (l-1),$$

and

$$Q (I\lambda^{l-1} + \lambda^{l-2}\Lambda + \cdots + \Lambda^{l-1}) R' = A_0^{-1}.$$

We may now put $\lambda = 0$ in these equations and obtain the results (23) and (22) respectively.

If A_l is non-singular, then equations (2) and (3) have no solution when $\lambda = 0$. That is, there are no null latent roots, and the matrix Λ^{-1} exists. We may therefore put $\lambda = 0$ in equation (25) (remembering the definition of (1)) and obtain the result (24). This completes the proof.

4. Ordinary differential equations with constant coefficients

Consider now the set of n simultaneous differential equations

$$D^{(l)}(\mathscr{D})\, \boldsymbol{q} \equiv (A_0 \mathscr{D}^l + A_1 \mathscr{D}^{l-1} + \cdots + A_{l-1}\mathscr{D} + A_l)\, \boldsymbol{q} = \boldsymbol{\xi}(t), \tag{26}$$

where $\mathscr{D}$ denotes the differential operator d/dt, and $\boldsymbol{q}$, $\boldsymbol{\xi}$, are vector functions of the independent variable t. We see immediately that if λ_i is a latent root of $D^{(l)}(\lambda)$ with associated latent vector $\boldsymbol{q}_i$, then $e^{\lambda_i t}\boldsymbol{q}_i$ is a solution of (26) when $\boldsymbol{\xi}(t) \equiv 0$. For

$$D^{(l)}(\mathscr{D})\, e^{\lambda_i t} \boldsymbol{q}_i = e^{\lambda_i t} D^{(l)}(\lambda_i)\, \boldsymbol{q}_i = 0.$$

Under the assumptions made throughout this paper there exist ln such solutions which are independent vector functions of t. The complementary function for the system of differential equations is therefore

$$\boldsymbol{q} = Q\, e^{\Lambda t} \boldsymbol{c}, \tag{27}$$

where $\boldsymbol{c}$ is a column vector of ln constants to be fixed by the boundary conditions. A particular integral of equations (26) can be written in the symbolic form

$$\boldsymbol{q} = [D^{(l)}(\mathscr{D})]^{-1} \boldsymbol{\xi}(t).$$

If the latent vectors are normalised according to (19), (20), or (21), then we may apply Theorem 2 with $r = 0$, and the particular integral becomes

$$\boldsymbol{q} = Q (I\mathscr{D} - \Lambda)^{-1} R' \boldsymbol{\xi}(t). \tag{28}$$

Hence

$$\boldsymbol{q} = Q\, e^{\Lambda t} \int_{t_0}^{t} e^{-\Lambda \tau} R' \boldsymbol{\xi}(\tau)\, d\tau, \tag{29}$$

where t_0 is an arbitrary constant. On combining this with (27), the lower limit of integration can be absorbed in the vector $\boldsymbol{c}$, and the complete solution can be written as

$$\boldsymbol{q} = Q\, e^{\Lambda t} \left\{ \boldsymbol{c} + \int^{t} e^{-\Lambda \tau} R' \boldsymbol{\xi}(\tau)\, d\tau \right\}. \tag{30}$$

If any of the first $(l-1)$ derivatives of the solution are required, we can use equation (9) to obtain

$$\mathscr{D}^r \boldsymbol{q}(t) = Q \Lambda^r e^{\Lambda t} \left\{ \boldsymbol{c} + \int^{t} e^{-\Lambda \tau} R' \boldsymbol{\xi}(\tau)\, d\tau \right\}, \tag{31}$$

for $r = 0, 1, 2, \ldots, (l-1)$.

5. Linear vibration problems

These problems often give rise to the equations

$$D^{(2)}(\mathscr{D})\, \boldsymbol{q} \equiv A_0 \ddot{\boldsymbol{q}} + A_1 \dot{\boldsymbol{q}} + A_2 \boldsymbol{q} = \boldsymbol{\xi}(t). \tag{32}$$

In this case the normalising conditions (20) and (21), on which Theorems 2 and 3 depend can be written

$$A_1^* + (\Lambda A_0^* + A_0^* \Lambda) = I_{2n}, \tag{33}$$

and

$$A_2^* - \Lambda A_0^* \Lambda = -\Lambda, \tag{34}$$

where $A_i^* = R' A_i Q$ are $(2n \times 2n)$ matrices. We then have from Theorem 3; $Q \Lambda R' = A_0^{-1}$, $Q R' = 0$, and if A_2 is non-singular, $Q \Lambda^{-1} R' = -A_2^{-1}$. A general solution of (32) is given by equation (30).

However, another interesting decomposition of $D^{(2)}(\lambda)$ is possible, and this gives rise to a rather different particular integral which can be obtained without normalising the latent vectors.

Divide the $2n$ latent roots into two sets of n roots, and form with these the $(n \times n)$ diagonal matrices Λ_1 and Λ_2, any set of multiple roots being contained entirely in Λ_1 or in Λ_2. Suppose that the corresponding latent vectors form the non-singular $(n \times n)$ matrices Q_1, Q_2 and R_1, R_2. Then in partitioned form,

$$Q = [Q_1, Q_2], \quad R = [R_1, R_2], \quad \text{and} \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}. \tag{35}$$

In equation (32) make the transformation to coordinates p_i given by

$$\boldsymbol{q} = Q_1 \boldsymbol{p},$$

and premultiply the resulting equation by R_2'. Then putting $R_2' \boldsymbol{\xi} = \boldsymbol{\zeta}$, we have

$$(\hat{A}_0 \mathscr{D}^2 + \hat{A}_1 \mathscr{D} + \hat{A}_2)\, \boldsymbol{p} = \boldsymbol{\zeta}, \tag{36}$$

where $\hat{A}_i = R_2' A_i Q_1$, for $i = 0, 1$, or 2.

The result we are now to obtain (equation (37)) is a decomposition, or factorisation, of the matrix operator in brackets in equation (36). The same sort of reasoning as that used in Sections 2 and 3 is required, but in order to avoid tedious repetitions I shall merely outline the argument.

If we refer back to Theorem 1, we see that equations similar to (5) can be established if we relax the normalisation condition, but replace the right-hand side of these equations by unspecified diagonal matrices (although (6) will no longer be true). The same is true of equations (19), (20) and (21).

If we now substitute from (35) via the definition of A_i^* into (33) and (34), supposing that the above normalisation has *not* been performed, then we obtain

$$\begin{bmatrix} R_1' \\ R_2' \end{bmatrix} A_1 [Q_1\, Q_2] + \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} R_1' \\ R_2' \end{bmatrix} A_0 [Q_1\, Q_2] + \begin{bmatrix} R_1' \\ R_2' \end{bmatrix} A_0 [Q_1\, Q_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$
$$= (\text{diagonal matrix}),$$

and

$$\begin{bmatrix} R_1' \\ R_2' \end{bmatrix} A_2 [Q_1\, Q_2] - \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} R_1' \\ R_2' \end{bmatrix} A_0 [Q_1\, Q_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} = (\text{diagonal matrix}).$$

If we pick out the partition in the (2, 1) position from each of these equations, we obtain

$$\hat{A}_1 = -(\Lambda_2 \hat{A}_0 + \hat{A}_0 \Lambda_1),$$

and

$$\hat{A}_2 = \Lambda_2 \hat{A}_0 \Lambda_1 .$$

Substituting these expressions in (36), we obtain

$$[\hat{A}_0 \mathscr{D}^2 - (\Lambda_2 \hat{A}_0 + \hat{A}_0 \Lambda_1)\, \mathscr{D} + \Lambda_2 \hat{A}_0 \Lambda_1]\, \boldsymbol{p} = \boldsymbol{\zeta},$$

and the left-hand side factorises to give

$$(I\mathscr{D} - \Lambda_2)\, \hat{A}_0 (I\mathscr{D} - \Lambda_1)\, \boldsymbol{p} = \boldsymbol{\zeta}. \tag{37}$$

Thus,

$$\begin{aligned} \boldsymbol{p} &= (I\mathscr{D} - \Lambda_1)^{-1} \hat{A}_0^{-1} (I\mathscr{D} - \Lambda_2)^{-1} \boldsymbol{\zeta}, \\ &= e^{\Lambda_1 t} \int_{t_0}^{t} e^{-\Lambda_1 \tau}\, \hat{A}_0^{-1}\, e^{\Lambda_2 \tau} \int_{t_0}^{\tau} e^{-\Lambda_2 \nu}\, \boldsymbol{\zeta}(\nu)\, d\nu\, d\tau . \end{aligned} \tag{38}$$

On inverting the order of integration the τ integral can be evaluated, and in terms of the unnormalised modes we obtain the particular integral:

$$\boldsymbol{q} = Q_1 \boldsymbol{p} = Q_1 \int_{t_0}^{t} [\beta\, e^{\Lambda_2 (t-\nu)} - e^{\Lambda_1 (t-\nu)} \beta]\, \boldsymbol{\zeta}(\nu)\, d\nu , \tag{39}$$

where β is the $(n \times n)$ matrix with elements $\alpha_{ij}/(\lambda_j - \lambda_i)$, α_{ij} are the elements of $\hat{A}_0^{-1}$, and λ_i, λ_j are the i^{th} and j^{th} diagonal elements of Λ_1 and Λ_2 respectively.

If $\boldsymbol{\xi}(t)$ is periodic, then it may be possible to represent it by a Fourier series. For such a case, and for the practical problem of steady sinusoidal motion occurring in resonance tests, it is important to consider the case in which

$$\boldsymbol{\xi}(t) = \boldsymbol{\xi}_0\, e^{i\omega t},$$

where $\boldsymbol{\xi}_0$ is independent of t. (Note that i is now $\sqrt{-1}$.) In these cases one is often interested in the particular integral part of the solution only; the complementary function then corresponds to a transient motion which eventually becomes negligible.

We can deduce the two following results from (29) and (39), or directly from (28) and (38). If (33) and (34) are satisfied, then

$$\boldsymbol{q} = Q(I i\omega - \Lambda)^{-1} R' \boldsymbol{\xi}_0\, e^{i\omega t}, \tag{40}$$

(for $i\omega \neq \lambda_j$). This result includes the special case considered by Gurtin [4], in which A_0, A_1, and A_2 are symmetric and all the latent roots are complex.

Otherwise, the above normalisation condition need not be fulfilled in order to use the result:

$$\boldsymbol{q} = Q_1 (I\omega + i\Lambda_1)^{-1} \hat{A}_0^{-1} (I\omega + i\Lambda_2)^{-1} R_2' \boldsymbol{\xi}_0 e^{i\omega t}. \tag{41}$$

The power of the results (40) and (41) lies in the fact that once Q, R, and Λ have been evaluated, $\boldsymbol{q}$ can very easily be calculated for a large number of values of ω. *I.e.* it is only necessary to invert *diagonal* matrices for each new ω to be considered (having once inverted $\hat{A}_0$ for (41)).

6. Significance of latent vectors

It is worthwhile to demonstrate that motion can occur in any one mode of free vibration $(\boldsymbol{\xi}(t) \equiv 0)$ without any tendency to disturb the system in the other modes. This is a well known property of conservative systems but does not appear to have been clearly stated for non-conservative systems. Such a mode of free vibration is described mathematically by a latent vector $\boldsymbol{q}_m$ of (2) with $l=2$. Physically, of course, we should only use the real part of $\boldsymbol{q}_m e^{\lambda_m t}$ to obtain the displacement of the system due to this mode at time t.

In order to demonstrate this we suppose that the system is first brought to rest and held in the position given by the real part of $\boldsymbol{q}_m$. It is then set in motion by a set of impulses which give rise to a velocity distribution given by the real part of $\lambda_m \boldsymbol{q}_m$. These one-point boundary conditions are to define the $(2n \times 1)$ column vector $\boldsymbol{c}$ of equation (27). On putting $t=0$ in equation (31) with $r=1$ and $r=0$, we obtain

$$\begin{bmatrix} Q\Lambda \\ Q \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} \lambda_m \boldsymbol{q}_m \\ \boldsymbol{q}_m \end{bmatrix},$$

which has the obvious solution in which $\boldsymbol{c}$ has a one in the m^{th} position and zeros elsewhere. On substituting this vector $\boldsymbol{c}$ in equation (30), we see that at any subsequent time the displacement of the system is given by the real part of

$$\boldsymbol{q} = \boldsymbol{q}_m e^{\lambda_m t}.$$

Acknowledgement. It is a pleasure to acknowledge my debt to Prof. A. M. Ostrowski for his valuable criticisms of the first draft of this paper. In particular, for his neat proof of Theorem 3, which I have presented above.

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Stability in the Large of Autonomous Systems of Two Differential Equations

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1. Introduction

In this paper[1] we discuss the stability in the large of some special systems of two differential equations with the help of qualitative methods (of course in combination with the construction of the Lyapunov function).

In Section 2, we consider the system of equations

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = f(\sigma) \quad \text{where} \quad \sigma = cx - dy$$

and obtain certain sufficient conditions which guarantee the stability in the large of the trivial solution of the above system.

In Section 3, we solve the problem of AIZERMAN for the following systems of two equations:

$$\begin{aligned} \frac{dx}{dt} &= ax + f(y) \\ \frac{dy}{dt} &= bx + cy \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{dx}{dt} &= f(x) + ay \\ \frac{dy}{dt} &= bx + cy. \end{aligned} \tag{2}$$

For system (1) MALKIN [*11*] showed that the trivial solution is asymptotically stable in the large under the conditions $a+c<0$, $\big(acy - bf(y)\big)\,y > 0$ for $y \neq 0$ and

$$\int_0^y \big(acy - bf(y)\big)\,dy \to +\infty \quad \text{as} \quad |y| \to +\infty. \tag{3}$$

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[1] This paper forms a part of my doctoral thesis submitted in the Department of Mathematics of the University of British Columbia, Vancouver, B. C., Canada. I express my thanks to Professor E. LEIMANIS for his help.

We prove a similar theorem without the requirement of (3). In the case of system (2) we give a new proof of a theorem which asserts that if $c^2+ab\neq 0$, then the trivial solution is asymptotically stable in the large under the generalized Hurwitz conditions. This theorem was first proved by ERUGIN.

In Section 4, we discuss the stability in the large of the systems

$$\frac{dx}{dt}=ax+f_1(y), \qquad \frac{dy}{dt}=f_2(x)+cy$$

$$\frac{dx}{dt}=f_1(x)+f_2(y), \qquad \frac{dy}{dt}=bx+cy.$$

2. Stability in the Large of the System $dx/dt=F(x,y)$, $dy/dt=f(\sigma)$

In this section we shall consider the system of equations

$$\begin{aligned}\frac{dx}{dt}&=F(x,y)\\ \frac{dy}{dt}&=f(\sigma), \qquad \sigma=cx-dy\end{aligned} \tag{2.1}$$

where c and d are constants, $c\neq 0$; the functions $F(x,y)$, $f(\sigma)$ are continuous and $F(0,0)=0$, $f(0)=0$. Besides, the fulfilment of conditions of uniqueness of the solution $x=0=y$ is assumed.

The above system was considered in the works of ERSHOV ([*3*], [*4*]) and KRASOVSKIĬ [*10*]. Following KRASOVSKIĬ, we transform the system (2.1) to the following form (2.2) by the change of dependent variables expressed by the relations

$$\sigma=cx-dy$$

$$y=y.$$

The above transformation is non-singular because c is assumed not equal to zero. We then have

$$\begin{aligned}\frac{d\sigma}{dt}&=\varphi(\sigma,y)\\ \frac{dy}{dt}&=f(\sigma)\end{aligned} \tag{2.2}$$

where

$$\varphi(\sigma,y)=cF\left(\frac{\sigma+dy}{c},y\right)-df(\sigma).$$

KRASOVSKIĬ constructed the following Lyapunov function for the system (2.2):

$$V(\sigma,y)=\int_0^\sigma f(\sigma)\,d\sigma-\int_0^y\varphi(0,y)\,dy,$$

and using Theorem 4 (BARBASHIN [*2*]) proved the following theorem:

Theorem 2.1. *If the conditions*

$$\sigma f(\sigma)>0 \qquad \text{for} \quad \sigma\neq 0 \tag{2.3}$$

$$y\varphi(0,y)<0 \qquad \text{for} \quad y\neq 0 \tag{2.4}$$

$$\sigma[\varphi(\sigma,y)-\varphi(0,y)]<0 \qquad \text{for} \quad \sigma\neq 0 \tag{2.5}$$

and
$$\left|\int_0^{\pm\infty} f(\sigma)\,d\sigma\right| = \infty, \qquad \left|\int_0^{\pm\infty} \varphi(0, y)\,dy\right| = \infty \tag{2.6}$$
are satisfied, then the trivial solution $x=0=y$ *of system* (2.1) *is asymptotically stable in the large.*

It may be remarked that conditions (2.3) and (2.4) can be replaced by the following conditions:

$$\sigma f(\sigma) < 0 \qquad \text{for} \quad \sigma \neq 0 \tag{2.3'}$$

$$y\varphi(0, y) > 0 \quad \text{for} \quad y \neq 0. \tag{2.4'}$$

Ershov [4] claimed that Theorem 2.1 holds without the requirement of conditions (2.6). In fact he stated the following theorem:

Theorem 2.2. *If conditions* (2.3), (2.4) *and* (2.5) *are satisfied for the system* (2.1), *then the trivial solution of system* (2.1) *is asymptotically stable in the large.*

The following example shows that conditions (2.6) cannot be removed in general.

Example. Consider the system of equations

$$\begin{aligned}\frac{dx}{dt} &= -y - f(x)\\ \frac{dy}{dt} &= f(x)\end{aligned} \tag{2.7}$$

where $f(x)$ is defined as below

$$\begin{aligned} f(x) &= \frac{e^{-2x}}{1+e^{-x}} \qquad \text{for} \quad x \geqq 1\\ &= \frac{e^{-2}}{1+e^{-1}}\,x \quad \text{for} \quad x < 1.\end{aligned}$$

Obviously, $xf(x) > 0$ for $x \neq 0$ and $f(0) = 0$

$$y\varphi(0, y) = y\left(-y - f(0)\right) = -y^2 < 0 \quad \text{for} \quad y \neq 0$$
$$\sigma[\varphi(\sigma, y) - \varphi(0, y)] = x\left(-y - f(x) + y\right) = -xf(x) < 0 \quad \text{for} \quad x \neq 0.$$

Moreover it is not difficult to show that $f(x)$ is continuous and satisfies the Lipschitz condition. Thus all the conditions of Theorem 2.2 are satisfied. We show that the trivial solution of this system is asymptotically stable in the sense of Lyapunov but not in the large. The stability in the small follows from the following Lyapunov function,

$$V(x, y) = \int_0^x f(x)\,dx + \tfrac{1}{2}y^2.$$

We now show that there exist trajectories going to infinity for $t \to +\infty$.

It is easy to verify that $y = -e^{-x}$ is a particular integral of the system on the interval $1 \leqq x < \infty$ passing through the point $(1, -e^{-1})$ at $t=0$. We show that along the curve $y = -e^{-x}$, $\frac{dx}{dt} > 0$, *i.e.*, x increases with the increase of time.

$$\frac{dx}{dt} = -y - f(x) = +e^{-x} - \frac{e^{-2x}}{1+e^{-x}} = \frac{e^{-x}}{1+e^{-x}} > 0 \quad \text{for} \quad x \geqq 1.$$

We integrate $\frac{dx}{dt} = \frac{e^{-x}}{1+e^{-x}}$ along the trajectory $y = -e^{-x}$ and have

$$\int_1^x \frac{1+e^{-x}}{e^{-x}}\, dx = \int_0^t dt \quad \text{or} \quad e^x + x\Big|_1^x = t\Big|_0^t \quad \text{or} \quad x + e^x - e - 1 = t.$$

From the last equation it follows that as $t \to +\infty$, $x \to +\infty$, *i.e.*, the positive half trajectory $y = -e^{-x}$ of the system (2.7) tends to infinity as $t \to +\infty$. Hence it follows that the trivial solution is not asymptotically stable in the large.

2.2. Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= f(\sigma), \qquad = cx - dy, \qquad c \neq 0 \end{aligned} \tag{2.2.1}$$

under the conditions:

$$F(0, 0) = 0, \qquad f(0) = 0 \tag{2.2.2}$$

$$\sigma f(\sigma) > 0 \quad \text{for} \quad \sigma \neq 0 \tag{2.2.3}$$

$$\sigma[\varphi(\sigma, y) - \varphi(0, y)] < 0 \quad \text{for} \quad \sigma \neq 0 \tag{2.2.4}$$

$$y\, \varphi(0, y) < 0 \quad \text{for} \quad y \neq 0 \tag{2.2.5}$$

where

$$\varphi(\sigma, y) = cF\left(\frac{\sigma + dy}{c}, y\right) - d f(\sigma). \tag{2.2.6}$$

Besides, the fulfilment of conditions of uniqueness of the solution $x = 0 = y$ is assumed. As before we reduce the system (2.2.1) to the following system:

$$\begin{aligned} \frac{d\sigma}{dt} &= \varphi(\sigma, y) \\ \frac{dy}{dt} &= f(\sigma). \end{aligned} \tag{2.2.7}$$

We consider the positions of the curves represented by the right-hand sides of (2.2.7) on the (σ, y) plane, *i.e.*, of $\varphi(\sigma, y) = 0$ and $f(\sigma) = 0$. Since $\sigma f(\sigma) > 0$ for $\sigma \neq 0$ and $f(0) = 0$, $f(\sigma) = 0$ only when $\sigma = 0$, *i.e.*, $f(\sigma) = 0$ represents the y-axis. We now turn to the curve represented by $\varphi(\sigma, y) = 0$. We observe that $\varphi(0, 0) = 0$ and on the σ-axis $\varphi(\sigma, y) > 0$ for $\sigma < 0$, $\varphi(\sigma, y) < 0$ for $\sigma > 0$; on the y-axis $\varphi(\sigma, y) > 0$ for $y < 0$, $\varphi(\sigma, y) < 0$ for $y > 0$. From these facts it follows that $\varphi(\sigma, y) < 0$, $\sigma > 0$, $y > 0$ and $\varphi(\sigma, y) > 0$ for $\sigma < 0$, $y < 0$. In deriving these conclusions we have made use of the conditions (2.2.4) and (2.2.5). Thus it follows that $\varphi(\sigma, y)$ changes sign in the second and fourth quadrants, and hence the curve $\varphi(\sigma, y) = 0$ lies in the second and fourth quadrants. It is easy to see that

$\frac{d\sigma}{dt} = \varphi(\sigma, y) > 0$ for the points lying to the left of the curve $\varphi(\sigma, y) = 0$

$\frac{d\sigma}{dt} = \varphi(\sigma, y) < 0$ for the points lying to the right of the curve $\varphi(\sigma, y) = 0$

$\frac{dy}{dt} = f(\sigma) > 0$ for the points lying to the right of y-axis

$\frac{dy}{dt} = f(\sigma) < 0$ for the points lying to the left of y-axis.

The curve $\varphi(\sigma, y)=0$ and the co-ordinate axes decompose the plane (σ, y) into six regions. $y(t)$ is maximum on the y-axis for $y>0$ and minimum for $y<0$; $\sigma(t)$ is maximum on the curve $\varphi(\sigma, y)=0$ for $\sigma>0$ and minimum for $\sigma<0$. The direction of motion is indicated in Fig. 1. We introduce polar co-ordinates

$$\sigma = r\cos\vartheta, \qquad y = r\sin\vartheta,$$

then

$$\dot{r} = \dot{\sigma}\cos\vartheta + \dot{y}\sin\vartheta$$

and

$$r\dot{\vartheta} = -\dot{\sigma}\sin\vartheta + \dot{y}\cos\vartheta.$$

The signs of $\dot{r}$ and $\dot{\vartheta}$ in different regions are given as below:

(1,4) $\dot{r}$ may be $\gtreqless 0$, $\quad \dot{\vartheta} > 0$

(2,5) $\dot{r}$ may be $\gtreqless 0$, $\quad \dot{\vartheta} > 0$

(3,6) $\dot{r} < 0$, $\quad \dot{\vartheta}$ may be $\gtreqless 0$.

The Lyapunov function for the system (2.2.7) is

$$V(\sigma, y) = \int_0^\sigma f(\sigma)\, d\sigma - \int_0^y \varphi(0, y)\, dy.$$

Its total time derivative in view of equations (2.2.7) is

$$\begin{aligned}\dot{V} &= f(\sigma)\,[\varphi(\sigma, y) - \varphi(0, y)] \\ &< 0 \quad \text{for} \quad \sigma \neq 0 \\ &= 0 \quad \text{for} \quad \sigma = 0.\end{aligned}$$

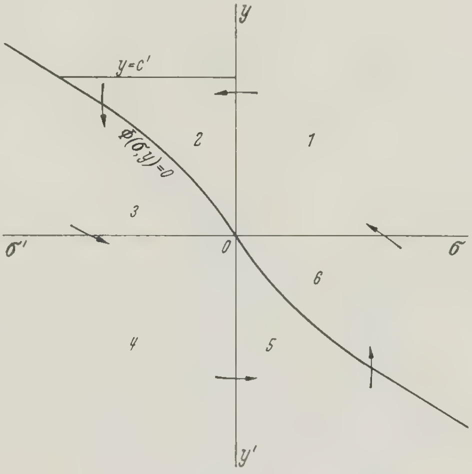

Fig. 1

Obviously, $V(\sigma, y)$ is positive definite and $\dot{V}$ is of negative sign. Now $\frac{d\sigma}{dt} = \varphi(0, y)$ for $\sigma=0$ and it is different from zero unless $y=0$. This means that $\sigma=0$ does not contain any other positive half trajectory of the system (2.2.7) than the origin. Thence follows the asymptotic stability in the sense of LYAPUNOV of the trivial solution of system (2.2.7). Thus we have shown

(i) the point (0, 0) is the only point of equilibrium,

(ii) the unperturbed motion is asymptotically stable according to LYAPUNOV,

(iii) there are no periodic motions, since for (2.2.7) a Lyapunov function has been constructed.

Since the σ positive half axis is intersected by the motions in one direction only, it can be taken for the straight line $L(0, \infty)$ appearing in Theorem 1.3 (ERUGIN [5]). Thus all the conditions of Theorem 1.3 (ERUGIN [5]) are satisfied except the fourth, *i.e.*, the motions having bounded polar angles are bounded. We now indicate what additional conditions are to be imposed in order to realize the fourth condition of Theorem 1.3 (ERUGIN [5]).

Let a motion $M(t)$ start in a region, say (6). Any motion $M(t)$ started in the region (6) or entering this region either tends to the origin or goes out of this region and enters the region (1). This follows from the fact that $\dot{r}<0$ in this region. After entering the region (1) the motion $M(t)$ either crosses the y-axis or tends to infinity along the y-axis, but it cannot go to the origin since $\dot{\vartheta}>0$. We are thus led to impose the following

Condition A. We assume that $\varphi(\sigma, y)$ satisfies such conditions called A that the motion entering or starting in the regions (1) and (4) leaves these regions with the increase of time.

This condition guarantees that there are no motions with bounded polar angles in the regions (1) and (4). We impose another condition called B.

Condition B. We assume that $\varphi(\sigma, c')=0$ has a solution for all c'.

If Condition A is satisfied, then the motion $M(t)$ enters the region (2). We now show that the motion $M(t)$ leaves the region (2) with the increase of time if Condition B is satisfied. In fact if $\varphi(\sigma, c')=0$ has a solution for all c', *i.e.*, if $y=c'$ intersects the curve $\varphi(\sigma, y)=0$, the motion after entering the region (2) can neither cross the line $y=c'$, since $\frac{dy}{dt}<0$ in this region go to the origin, because $\dot{\vartheta}>0$ in region (2). Therefore the motion must intersect the curve $\varphi(\sigma, y)=0$. Similar reasoning can be carried out in the regions (3), (4) and (5). The above analysis shows that there are no motions with bounded polar angles in the regions (1), (2), (4) and (5) and that the motions with bounded polar angles can occur only in the regions (3) and (6) and as proved above they are bounded. Thus any motion with bounded polar angle is bounded.

Condition B was imposed with a view to ensure that in the regions (2) and (5) there are no motions with bounded polar angles. This can also be achieved if we assume that in these regions $\dot{\vartheta}>\varepsilon>0$. This condition we call *Condition C*.

We now collect these results in the following

Theorem 2.2.1. *If Conditions* (2.2.2)—(2.2.5) *are satisfied and if either Conditions A and B or Conditions A and C are satisfied, then the trivial solution of* (2.2.1) *is asymptotically stable in the large.*

It may be remarked that Conditions B and C are independent, *i.e.*, if Condition B holds then Condition C may or may not hold or vice versa. In the linear case, *i.e.*, when the right-hand sides of system (2.2.1) are linear, however, both hold.

Let us consider a few examples now.

Example 1. Consider the system

$$\frac{dx}{dt} = -y + e^{-x} - 1$$
$$\frac{dy}{dt} = x. \tag{2.2.8}$$

Obviously, this system satisfies the conditions (2.2.2)—(2.2.5). We verify that in this case Condition A is satisfied. Let us for the sake of definiteness assume that the motion $M(t)$ is in the region (1). We assume that the Condition A is not satisfied. Then the motion $M(t)$ goes to infinity as $t\to+\infty$ and during this $y(t)\to+\infty$. From the first equation of (2.2.8) follows that dx/dt becomes infinitely large but negative, which means that $M(t)$ cannot remain in the first quadrant, and hence our assumption is not true. It is not difficult to verify that the straight line $y=c$ does not intersect the curve

$$-y+e^{-x}-1=0 \qquad \text{for all } c \text{ (see Fig. 2).}$$

Hence, the Condition B is not satisfied. However, the Condition C holds. In fact $r\dot{\vartheta}=-\dot{x}\sin\vartheta+\dot{y}\cos\vartheta=-\sin\vartheta(-y+e^{-x}-1)+x\cos\vartheta$ and

$$\dot{\vartheta}=-\frac{\sin\vartheta}{r}(-y+e^{-x}-1)+\cos^2\vartheta>\varepsilon>0$$

in the regions (2) and (5). This establishes the asymptotic stability in the large of the trivial solution of (2.2.8).

Example 2. Consider the equation

$$\frac{d^2y}{dt^2}+\varphi\left(\frac{dy}{dt}\right)g(y)+f(y)=0. \quad (2.2.9)$$

This equation can be thrown into the system

$$\frac{dx}{dt}=-\varphi(x)\,g(y)-f(y)$$
$$\frac{dy}{dt}=x$$

by writing x for dy/dt.

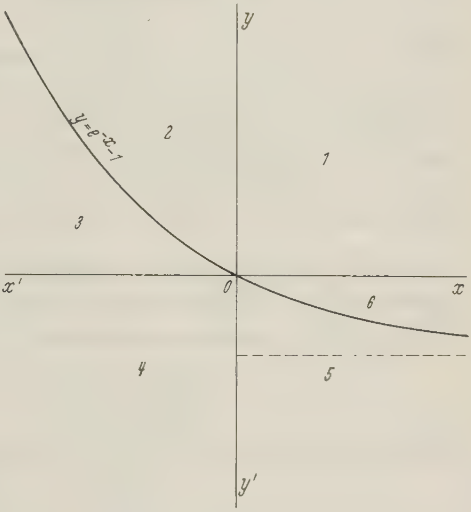

Fig. 2

We assume that the following conditions are fulfilled:

$$y\,f(y)>0, \qquad f(0)=0$$

and either

$$h_1(x)>0 \quad \text{for} \quad x\neq 0 \quad \text{and} \quad g(y)>0 \quad \text{for all } y$$

or

$$h_1(x)<0 \quad \text{for} \quad x\neq 0 \quad \text{and} \quad g(y)<0 \quad \text{for all } y$$

where

$$h_1(x)=\frac{\varphi(x)}{x} \quad \text{for} \quad x\neq 0.$$

The Lyapunov function for this system is

$$V(x,y)=\int_0^x x\,dx+\int_0^y f(y)\,dy.$$

Then

$$\dot{V}=x\left(-\varphi(x)\,g(y)-f(y)\right)+f(y)\,x$$
$$=-x\,\varphi(x)\,g(y).$$

It is not difficult to see that in this case Condition C of Theorem 2.2.1 is satisfied. Thus if Condition A is also satisfied then the trivial solution of (2.2.9) is asymptotically stable in the large.

Example 3. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= \varphi(x) + f_2(y) \\ \frac{dy}{dt} &= f_1(x). \end{aligned} \tag{2.2.10}$$

For the system (2.2.10) the V function is

$$V(x, y) = \int_0^x f_1(x)\,dx - \int_0^y f_2(y)\,dy.$$

Then

$$V(x, y) = f_1(x)\,\varphi(x).$$

We subject (2.2.10) to the following conditions:

$$\begin{gathered} \varphi(0) = 0, \quad h_3(x) < 0 \quad \text{for} \quad x \neq 0, \quad \text{where} \quad h_3(x) = \frac{\varphi(x)}{x} \quad \text{for} \quad x \neq 0 \\ f_1(0) = 0 \text{ and } x f_1(x) > 0 \text{ for } x \neq 0, \quad f_2(0) = 0 \text{ and } y f_2(y) < 0 \text{ for } y \neq 0. \end{gathered} \tag{2.2.11}$$

The trivial solution of (2.2.10) is thus asymptotically stable in the large if in addition to (2.2.11) the Conditions A and B or A and C are satisfied.

Example 4. Consider the equation of the second order

$$\frac{d^2y}{dt^2} + f\left(\frac{dy}{dt}, y\right)\frac{dy}{dt} + g(y) = 0. \tag{2.2.12}$$

We write it in the form of a system of equations

$$\begin{aligned} \frac{dx}{dt} &= -f(x, y)\,x - g(y) \\ \frac{dy}{dt} &= x. \end{aligned}$$

We assume that

$$g(0) = 0, \quad y\,g(y) > 0 \quad \text{for} \quad y \neq 0$$

and

$$f(x, y) > 0 \quad \text{for} \quad x \neq 0.$$

It is easy to verify that asymptotic stability in the large holds if Condition A is satisfied.

3. The Problem of Aĭzerman

In 1949 AĬZERMAN [*1*] proposed the following problem.

Let there be given a system of linear differential equations

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}\,x_j + a\,x_k, \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}\,x_j \qquad (i = 2, 3, \ldots, n). \tag{3.1}$$

Suppose that for the given constants a_{ij} $(i=1, \ldots, n,\ j=1, \ldots, n)$ and for an arbitrary value of a from the interval $\alpha < a < \beta$ all the roots of the characteristic equation of (3.1) have negative real parts. Let $a\,x_k$ be replaced by $f(x_k)$ in (3.1). We then have

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{ij}\,x_j + f(x_k), \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}\,x_j \qquad (i = 2, \ldots, n). \tag{3.2}$$

It is required to find out whether the trivial solution $x_1 = x_2 = \cdots = x_n = 0$ of system (3.2) is asymptotically stable in the large or not, for an arbitrary choice of the continuous function $f(x_k)$, which reduces to zero for $x_k = 0$ and which satisfies the inequality

$$\alpha x_k^2 < x_k f(x_k) < \beta x_k^2 \quad \text{for} \quad x_k \neq 0. \tag{3.3}$$

The answer to the above problem is in the negative (PLISS [*12*]). The interest in the problem is revived if we slightly change the above problem and ask ourselves the following questions. For what values of a_{ij} is the answer to the above problem in the affirmative and for what values in the negative? If the trivial solution is not asymptotically stable in the large under the generalized Hurwitz conditions (*i.e.*, the condition (3.3) and $f(0) = 0$), what additional assumptions should be made on $f(x_k)$ so that the trivial solution becomes asymptotically stable in the large? We shall discuss here the case when $n = 2$. The case $n = 3$, $k = 2$ has been completely solved recently by PLISS [*13*]. First of all we take $n = 2 = k$, *i.e.*, we consider the system

$$\begin{aligned} \frac{dx}{dt} &= a x + f(y) \\ \frac{dy}{dt} &= b x + c y \end{aligned} \tag{3.4}$$

under the assumptions that

$$f(0) = 0, \quad a + c < 0, \quad a c - b h(y) > 0, \quad y \neq 0 \tag{3.5}$$

where

$$h(y) = \frac{f(y)}{y} \quad \text{for} \quad y \neq 0.$$

Besides, the uniqueness of the trivial solution is assumed. System (3.4) was considered by N. P. ERUGIN [*6*] and I. G. MALKIN [*11*].

MALKIN [*11*] showed that the trivial solution of system (3.4) is asymptotically stable in the large if for sufficiently large values of $|y|$ the inequality $a c - b h(y) > \varepsilon$ holds. This condition can be relaxed to the condition that

$$\int_0^y \left(a c - b h(y)\, y\right) dy \to +\infty \quad \text{as} \quad |y| \to +\infty. \tag{3.6}$$

We show that asymptotic stability in the large of the trivial solution of (3.4) holds without the requirement of condition (3.6). For $b = 0$, from (3.5) follows that $a < 0$, $c < 0$, and an immediate integration of the system (3.4) shows that the equilibrium is asymptotically stable for arbitrary initial disturbances and for an arbitrary choice of the function $f(y)$.

Let $b \neq 0$. We introduce new variables defined by

$$\begin{aligned} x' &= b x - a y \\ y' &= y; \end{aligned}$$

then

$$\begin{aligned} \frac{dx'}{dt} &= b\frac{dx}{dt} - a\frac{dy}{dt} = b\left(a x + f(y)\right) - a(b x + c y) = -y'\left(a c - b h(y')\right) \\ \frac{dy'}{dt} &= \frac{dy}{dt} = b x + c y = x' + a y' + c y' = x' + (a + c)\, y'. \end{aligned}$$

Thus the system (3.4) reduces to the following:

$$\begin{aligned} \frac{dx}{dt} &= -y\big(ac - b\,h(y)\big) \\ \frac{dy}{dt} &= x + (a+c)\,y. \end{aligned} \tag{3.7}$$

We represent the curves $-y\big(ac-bh(y)\big)=0$ and $x+(a+c)y=0$ obtained by putting the right-hand sides of (3.7) equal to zero on the (x, y) plane. The first of these represents the straight line $y=0$, and the second the straight line $y=-\frac{x}{a+c}$. It is easy to see that

$$\frac{dx}{dt} = -y\big(ac - b\,h(y)\big) > 0 \quad \text{below the } x\text{-axis}$$

$$\frac{dx}{dt} = -y\big(ac - b\,h(y)\big) < 0 \quad \text{above the } x\text{-axis}$$

$$\frac{dy}{dt} = x + (a+c)\,y > 0 \quad \text{below the straight line} \quad x + (a+c)\,y = 0$$

$$\frac{dy}{dt} = x + (a+c)\,y < 0 \quad \text{above the straight line} \quad x + (a+c)\,y = 0.$$

The direction of motion is indicated in Fig. 3. The straight line $x+(a+c)y=0$ and the co-ordinate axes divide the plane (x, y) into six regions. We introduce polar co-ordinates $x=r\cos\varphi$, $y=r\sin\varphi$. Then $\dot r=\dot x\cos\varphi+\dot y\sin\varphi$ and $r\dot\varphi=-\dot x\sin\varphi+\dot y\cos\varphi$. It is not difficult to verify that in different regions the signs of $\dot r$ and $\dot\varphi$ are given as below: (1,4) $\dot r$ may be $\gtreqless 0$, $\dot\varphi>0$; (2,5) $\dot r<0$, $\dot\varphi$ may be $\gtreqless 0$; (3,6) $\dot r$ may be $\gtreqless 0$, $\dot\varphi>0$.

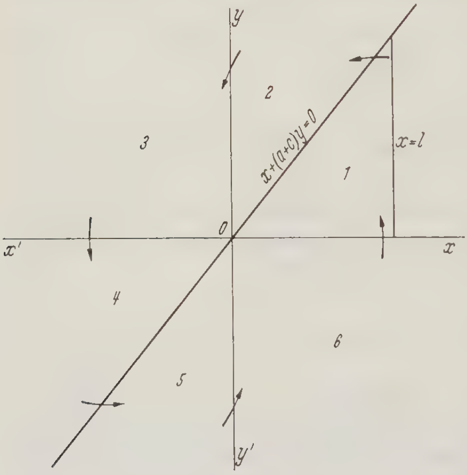

Fig. 3

The motion started in the region (6) must cross the axis of x. This follows from the inequality

$$\frac{dy}{dt} = x + (a+c)\,y \geqq d > 0$$

in the region (6). After entering the region (1), it cannot remain there and must cross the straight line $x+(a+c)y=0$ with the increase of time, since $x=l$ intersects the straight line $x+(a+c)y=0$ for all l. The Lyapunov function for the system (3.7) is easily seen to be

$$2V = x^2 + 2\int_0^y \big(ac - b\,h(y)\big)\,y\,dy.$$

Repeating the same argument as in Theorem 2.2.1, we arrive at the following

Theorem 3.1. *The trivial solution of system* (3.4) *is asymptotically stable in the large under Conditions* (3.5).

It may be noted that we may or may not have $\dot{\varphi} > \varepsilon > 0$ in the regions (1) and (4). Before studying the case $n=2$, $k=1$, we discuss the system

$$\begin{aligned} \frac{dx}{dt} &= y - F(x) \\ \frac{dy}{dt} &= -g(x). \end{aligned} \tag{3.8}$$

This system is a particular case of the system (2.2.1). Since it is quite an important system, we discuss it independently. The conditions to which (3.8) is subjected are

$$\begin{aligned} &xF(x) > 0, \\ &xg(x) > 0 \quad \text{for} \quad x \neq 0 \\ &F(0) = 0 = g(0). \end{aligned} \tag{3.9}$$

The V-function in this case is

$$2V = y^2 + 2\int_0^x g(x)\,dx \tag{3.10}$$

and

$$\begin{aligned} \frac{dV}{dt} &= y(-g(x)) + g(x)\,(y - F(x)) \\ &= -g(x)F(x) < 0 \quad \text{for} \quad x \neq 0 \\ &= 0 \quad \text{for} \quad x = 0. \end{aligned}$$

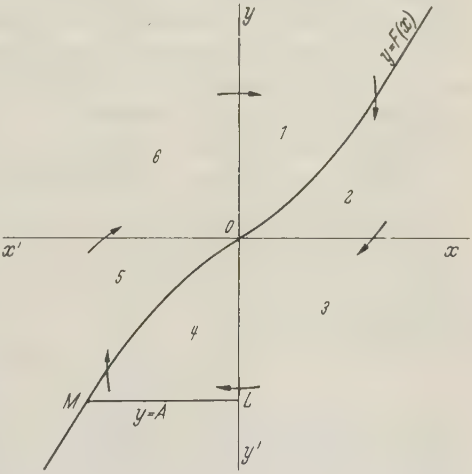

Fig. 4

The straight line $x=0$ does not contain a positive half trajectory of the system (3.8) except the origin. In fact, $dx/dt = y$ for $x=0$. Hence it follows that the trivial solution is asymptotically stable in the small. Since there exists a Lyapunov function V for the system, there cannot be any limit cycle. Also we note that the origin is the only point of equilibrium. It is not difficult to see in virtue of the Conditions (3.9) that

$$\frac{dx}{dt} = y - F(x) > 0 \quad \text{to the left of the curve } y = F(x)$$

$$\frac{dx}{dt} = y - F(x) < 0 \quad \text{to the right of the curve } y = F(x)$$

$$\frac{dy}{dt} = -g(x) > 0 \quad \text{to the left of the } y\text{-axis}$$

$$\frac{dy}{dt} = -g(x) < 0 \quad \text{to the right of the } y\text{-axis.}$$

The curve $y = F(x)$ and the co-ordinate axes divide the plane (x, y) into six regions. The direction of motion is represented in Fig. 4. We introduce polar co-ordinates $x = r\cos\varphi$, $y = r\sin\varphi$. Then

$$\dot{r} = \dot{x}\cos\varphi + \dot{y}\sin\varphi$$

$$r\dot{\varphi} = -\dot{x}\sin\varphi + \dot{y}\cos\varphi.$$

The signs of $\dot{r}$ and $\dot{\varphi}$ in the regions are given as below: (1,4) $\dot{r}$ may be $\gtreqless 0$, $\dot{\varphi} < 0$; (2,5) $\dot{r} < 0$, $\dot{\varphi}$ may be $\gtreqless 0$; (3,6) $\dot{r}$ may be $\gtreqless 0$, $\dot{\varphi} < 0$.

Let us follow the motion $M(t)$ after it intersects the negative half y-axis. Let us assume that $y=A$ intersects the curve $y=F(x)$ for all A, *i.e.*, $F(x)=A$ has a solution for all A; then, since in the region (4) $\dot{\varphi}<0$ and y is increasing, the motion $M(t)$ cannot remain in the bounded region OLM. It definitely cannot cross LM and OL. Therefore it must cross the curve $y=F(x)$ and enter the region (5). In the region (5) it either enters the origin or the region (6). This follows from the fact that $\dot{r}<0$ in this region. Since $dx/dt=-F(x)+y\geqq d>0$ in the region (6), the motion must leave this region and enter the region (1). Similar argument holds for the regions (1), (2) and (3). This shows that there are no motions with bounded polar angles in the regions (1), (3), (4) and (6). The motions with bounded polar angles can only occur in the regions (2) and (5), and as proved above they are bounded. Thus it follows that all motions with bounded polar angles are bounded. For the straight line $L(0,\infty)$ can be taken the positive half y-axis. Thus all the conditions of Theorem 1.3 (ERUGIN [5]) are satisfied, and we have the following theorem:

Theorem 3.2. *If the Conditions* (3.9) *are satisfied and* $F(x)=A$ *has a solution for all* A, *then the trivial solution of* (3.8) *is asymptotically stable in the large.*

We remark that by using Theorem 4 (BARBASHIN [2]) the stability in the large holds if

$$\int_0^x g(x)\,dx\to\infty \quad \text{as} \quad |x|\to\infty.$$

This is required in order to make the V function in (3.10) infinitely large. We further remark that the boundedness of the solution in Theorems 3.1 and 3.2 could be shown by making use of Lyapunov functions V, as has been suggested by Professor J. P. LA SALLE.

The widely discussed equation

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0 \tag{3.11}$$

can be dealt in the same way as the system (3.8), since (3.11) can be transformed to

$$\frac{dx}{dt}=y-F(x)$$

$$\frac{dy}{dt}=-g(x) \quad \text{where} \quad F(x)=\int_0^x f(x)\,dx.$$

We shall now discuss the Aĭzerman problem for the case $n=2$, $k=1$, *i.e.*, the system

$$\begin{aligned}\frac{dx}{dt}&=f(x)+ay\\ \frac{dy}{dt}&=bx+cy\end{aligned} \tag{3.12}$$

under the conditions

$$f(0)=0; \quad c+h(x)<0, \quad ch(x)-ab>0 \quad \text{for} \quad x\neq 0 \tag{3.13}$$

where

$$h(x)=\frac{f(x)}{x}, \quad x\neq 0.$$

This system is discussed in the works of Erugin ([5], [7]) and Malkin [11]. It was proved by Erugin that if $c^2+ab\neq 0$, then the trivial solution of (3.12) is asymptotically stable in the large under conditions (3.13). The proof by Erugin of this theorem is quite lengthy, so we give here a short proof of the theorem. It will become clear from our proof why the stability in the large does not hold under conditions (3.13) in the case $c^2+ab=0$. Erugin also showed that for the stability in the large in the case $c^2+ab=0$, it is sufficient that

$$\begin{aligned}\overline{\lim_{x\to\infty}}\left[\int_0^x \big(c f(x)-a b x\big)\,dx + c f(x) - a b x\right] &= +\infty\\ \overline{\lim_{x\to-\infty}}\left[\int_0^x \big(c f(x)-a b x\big)\,dx - c f(x) + a b x\right] &= +\infty.\end{aligned} \tag{3.14}$$

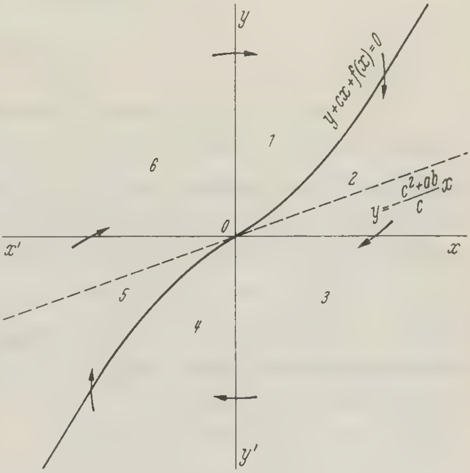

Fig. 5

Krasovskiĭ [8] showed that conditions (3.14) are necessary as well as sufficient. The question of the region of stability in the case where the stability in the large does not hold is discussed by Pliss [12].

We assume that $c^2+ab\neq 0$. Let $a=0$; then immediate integration of the system yields the stability in the large of the trivial solution $x=0=y$ of system (3.12). Now let $a\neq 0$. We introduce new dependent variables

$$\begin{aligned}x' &= x\\ y' &= a y - c x.\end{aligned}$$

Then

$$\begin{aligned}\frac{dx'}{dt} &= \frac{dx}{dt} = f(x') + y' + c x'\\ \frac{dy'}{dt} &= a\frac{dy}{dt} - c\frac{dx}{dt} = a(b x + c x) - c\big(f(x) + a y\big)\\ &= a b x' - c f(x').\end{aligned}$$

The system (3.12) is reduced to the form

$$\begin{aligned}\frac{dx}{dt} &= f(x) + c x + y\\ \frac{dy}{dt} &= -x\big(c h(x) - a b\big).\end{aligned}$$

Comparing it with system (3.8), we have

$$F(x) = -x\big(c + h(x)\big) \quad\text{and}\quad g(x) = x\big(c h(x) - a b\big).$$

The curve $y+cx+f(x)=0$ and the co-ordinate axes divide the (x, y) plane into six regions. The direction of motion is represented in Fig. 5. We consider the following cases.

Case i. $c<0$. Consider the straight line

$$y=-\frac{c^2+ab}{c}x. \tag{3.15}$$

If $c^2+ab>0$, then the straight line (3.15) lies in the first and third quadrants. Let us see how it is situated on the (x, y) plane with respect to the curve

$$y+cx+f(x)=0. \tag{3.16}$$

Let $y^{(1)}$ and $y^{(2)}$ denote the ordinates of (3.16) and (3.15) respectively; then

$$\begin{aligned} y^{(1)}-y^{(2)} &= -cx-f(x)+\frac{c^2+ab}{c}x \\ &= \frac{-c^2x-cf(x)+(c^2+ab)x}{c}=\frac{-x(ch(x)-ab)}{c}\gtrless 0 \end{aligned}$$

$$\text{according as } x\gtrless 0$$

i.e., the curve (3.16) lies above the straight line (3.15) in the first quadrant and below in the third quadrant. Obviously, the straight lines $y=A$ and $y=-\frac{c^2+ab}{c}x$ intersect, whence follows that $y=A$ and the curve $y=-cx-f(x)$ intersect for all A. Hence the trivial solution of (3.12) is asymptotically stable in the large in this case according to Theorem 3.2.

If $c^2+ab<0$, then the straight line $y=-\frac{c^2+ab}{c}x$ lies in the second and fourth quadrants, and no such conclusion as above can be drawn. However we can prove that in this case $\dot{\varphi}<-\varepsilon<0$ in the regions (1) and (4) which ensures the stability in the large.

According to Conditions (3.13)

$$c+h(x)<0, \quad ch(x)-ab>0 \quad \text{for} \quad x\neq 0$$

i.e.

$$h(x)<-c, \quad h(x)<\frac{ab}{c}$$

but since $c^2+ab<0$, the above inequalities are satisfied if we take $h(x)<-c$.

$f(x)$ can then be written as

$f(x)=-cx-\alpha(x)$ where $\alpha(x)\gtrless 0$ according as

$$x\gtrless 0.$$

Let us calculate $\dot{\varphi}$:

$$\begin{aligned} r\dot{\varphi} &= -\sin\varphi\,(f(x)+cx+y)-\cos\varphi\,(ch(x)-ab)\,x \\ &= -\sin\varphi\,(y-\alpha(x))-\cos\varphi\,[c(-cx-\alpha(x))-abx] \\ &= -\sin\varphi\,(y-\alpha(x))+\cos\varphi\,[(c^2+ab)\,x+c\alpha(x)] \\ \dot{\varphi} &= -\frac{\sin\varphi}{r}(y-\alpha(x))+\frac{\cos\varphi}{r}[(c^2+ab)\,x+c\alpha(x)]<-\varepsilon<0 \end{aligned}$$

in the regions (1) and (4).

Case ii. $c>0$. In this case $c^2+ab<0$, because from Conditions (3.13)

$$\frac{ab}{c} < h(x) < -c.$$

Let $h(x)$ be taken as $h(x)=-c-\alpha(x)$. Then

$$\frac{ab}{c} < -c-\alpha(x) < -c$$

$$c+\frac{ab}{c} < -\alpha(x) < 0$$

$$-\frac{c^2+ab}{c} > \alpha(x) > 0$$

or

$$0<\alpha(x)<-\frac{c^2+ab}{c}.$$

Let us consider the region (1). If $\dot{\varphi} \nless -\varepsilon<0$, then

$$\varliminf_{x\to\infty} \frac{-1}{x} \left[(c^2+ab)\,x + c\,x\,\alpha(x)\right] = 0, \qquad \textit{i.e.},$$

$\varlimsup\limits_{x\to\infty} x\alpha(x)=\infty$, whence it follows that the straight line $y=A$ intersects the curve

$$y=-c\,x-f(x) \quad \text{for all } A.$$

In fact,

$$-A=c\,x+f(x)=c\,x+\left(-c\,x-x\,\alpha(x)\right)=-x\,\alpha(x);$$

therefore $x\,\alpha(x)=A$. If $x\,\alpha(x)=A$ does not have a solution for all A, then

$$\varlimsup_{x\to\infty} x\,\alpha(x) = \text{finite} = D$$

$$\begin{aligned}\varliminf_{x\to\infty} -\left[(c^2+ab)\,x+c\,x\,\alpha(x)\right] &\geqq \varliminf_{x\to\infty}\left[-(c^2+ab)\,x\right]+\varliminf_{x\to\infty}\left(-c\,x\,\alpha(x)\right)\\ &= \varliminf_{x\to\infty} -x(c^2+ab)-\varlimsup_{x\to\infty}\left(c\,x\,\alpha(x)\right),\end{aligned}$$

whence follows that $\dot{\varphi}<-\varepsilon<0$.

Case iii. $c=0$. The System (3.12) reduces to

$$\frac{dx}{dt}=f(x)+y$$

$$\frac{dy}{dt}=a\,b\,x,$$

and the Conditions (3.13) reduce to $h(x)<0$, $ab<0$.

For this system we take the following Lyapunov function

$$2V=-a\,b\,x^2+y^2,$$

the derivative of which is

$$\frac{dV}{dt}=-a\,b\,x^2\,h(x).$$

This V function satisfies all the conditions of Theorem 4 (BARBASHIN [2]). Combining all these results, we have the following theorem:

Theorem 3.3. *If for the System* (3.12) $c^2 + ab \neq 0$, *then the trivial solution is asymptotically stable in the large under Conditions* (3.13).

In the case $c^2 + ab = 0$, we remark that $c < 0$ and from the proof of stability in the Case i, we find that the straight line $y = -\frac{c^2 + ab}{c} x$ coincides with the x-axis, and hence we can neither say that the straight line $y = A$ intersects the curve $y + cx + f(x) = 0$ for all A nor $\dot{\varphi} < -\varepsilon < 0$ in the regions (1) and (4).

4. A Generalization of the Problem of Aǐzerman

In this section we discuss the system of differential equations given by

$$\begin{aligned} \frac{dx}{dt} &= a x + f_1(y) \\ \frac{dy}{dt} &= f_2(x) + c y \end{aligned} \tag{4.1}$$

under the conditions

$$\begin{gathered} a + c < 0, \quad a c - h_1(y)\, h_2(x) > 0 \quad \text{for} \quad x \neq 0, \quad y \neq 0, \quad f_1(0) = 0 \\ = f_2(0) = 0 \end{gathered} \tag{4.2}$$

where $h_1(y)$ and $h_2(x)$ are defined by $f_1(y) = y\, h_1(y)$ and $f_2(x) = x\, h_2(x)$.

The above system was first discussed by KRASOVSKIǏ [*9*] who obtained certain theorems regarding the stability in the large of the trivial solution $x = 0 = y$ of (4.1). We prove here the following theorem:

Theorem 4.1. *If either* $h_1(y) > 0$ *for* $y \neq 0$ *and* $h_2(x) > 0$ *for* $x \neq 0$ *or* $h_1(y) < 0$ *for* $y \neq 0$, $h_2(x) < 0$ *for* $x \neq 0$, *then the trivial solution of* (4.1) *is asymptotically stable in the large under Conditions* (4.2).

Proof. We assume $h_1(y) > 0$, $y \neq 0$, and $h_2(x) > 0$, $x \neq 0$. From Conditions (4.2) follows that $a < 0$, $c < 0$. We consider the curves

$$a x + f_1(y) = 0 \quad \text{and} \quad f_2(x) + c y = 0. \tag{4.3}$$

Since $h_1(y) > 0$, $y \neq 0$ and $a < 0$, the curve $a x + f_1(y) = 0$ lies in the first and third quadrants. The same is true for the curve $f_2(x) + c y = 0$. We further note that the curve $a x + f_1(y) = 0$ lies above the curve $f_2(x) + c y = 0$ in the first quadrant and below in the third quadrant. In fact, if y_1 and y_2 denote the ordinates of the curves in (4.3) respectively, then

$$y_1 - y_2 = \frac{-a x}{h_1(y_1)} + \frac{x\, h_2(x)}{c} = -x \frac{a c - h_1(y_1)\, h_2(x)}{c\, h_1(y_1)} \gtrless 0 \quad \text{according as} \quad x \gtrless 0.$$

It is easy to see that

$$\begin{aligned} \frac{dx}{dt} &= a x + f_1(y) > 0 && \text{to the left of the curve} && a x + f_1(y) = 0 \\ \frac{dx}{dt} &= a x + f_1(y) < 0 && \text{to the right of the curve} && a x + f_1(y) = 0 \\ \frac{dy}{dt} &= f_2(x) + c y < 0 && \text{to the left of the curve} && c y + f_2(x) = 0 \\ \frac{dy}{dt} &= f_2(x) + c y > 0 && \text{to the right of the curve} && c y + f_2(x) = 0. \end{aligned}$$

The function $x(t)$ is maximized on the curve $ax+f_1(y)=0$ for $y>0$ and minimized for $y<0$; and the function $y(t)$ is maximized on the curve $f_2(x)+cy=0$ for $y>0$ and minimized for $y<0$. The curves and the direction of motion are shown in Fig. 6. As before we introduce polar co-ordinates $x=r\cos\varphi$, $y=r\sin\varphi$. Then $\dot r=\dot x\cos\varphi+\dot y\sin\varphi$,

$$r\dot\varphi=-\dot x\sin\varphi+\dot y\cos\varphi.$$

The signs of $\dot r$ and $\dot\varphi$ in different regions are (1,5) $\dot r$ may be $\gtreqless 0$, $\dot\varphi<0$ (2,6) $\dot r<0$, $\dot\varphi$ may be $\gtreqless 0$ (3,7) $\dot r$ may be $\gtreqless 0$, $\dot\varphi>0$ (4,8) $\dot r<0$, $\dot\varphi$ may be $\gtreqless 0$.

From Theorem 2.1 of Erugin's work [5] it follows that there is at least one integral curve going to the origin, the only point of equilibrium, in each of the regions (4) and (8). Other motions started in the regions (4) or (8) either go to the origin or enter the regions (3), (5) or (1), (8) (since $\dot r<0$, $\dot\varphi$ may be $\gtreqless 0$ in (4) and (8)). Let us suppose that the motion enters the region (1). We show that the motion leaves this region with the increase of time. Let c_2 be the least upper bound of $h_2(x)$. c_2 is finite, otherwise $ac-h_1(y)\,h_2(x)>0$ will not hold. We consider the straight line $c_2x+cy=0$. It is easy to verify that this straight line lies above the curve $f_2(x)+cy=0$ and below the curve $ax+f_1(y)=0$ in the first quadrant. In the third quadrant the positions are reversed. We consider the region bounded by $y=l$, the curve $ax+f_1(y)=0$ and the positive half y-axis. The motion cannot intersect the straight line $y=l$ since $\frac{dy}{dt}<0$. It cannot go to the origin since $\dot\varphi<0$. Therefore it must intersect the curve $ax+f_1(y)=0$ and enter the region (2) where it goes to the origin with the increase of time. For region (3) we take the straight line $x=l'$ and similarly show that the motion crosses the curve $f_2(x)+cy=0$ and enters the region (2) where it goes to the origin as $t\to+\infty$. Similar reasoning holds for the regions (5) and (7), which completes the proof of Theorem 4.1.

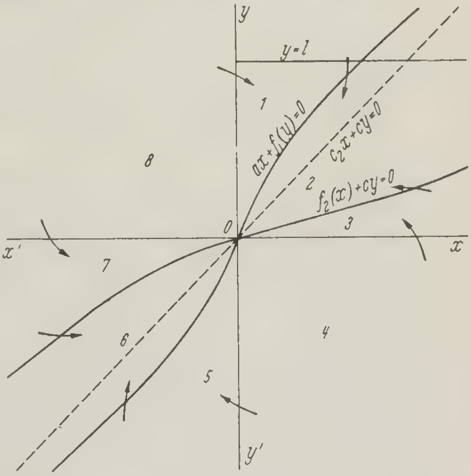

Fig. 6

In the same way we can prove the following theorem for the system

$$\begin{aligned}\frac{dx}{dt}&=f_1(x)+f_2(y)\\ \frac{dy}{dt}&=bx+cy\end{aligned}\qquad(4.4)$$

under the conditions

$$h_1(x)+c<0,\quad x\neq 0,\quad c\,h_1(x)-h_2(y)\,b>0,\quad x\neq 0,\quad y\neq 0,$$

$$f_1(0)=f_2(0)=0$$

where

$$h_1(x) = \frac{f_1(x)}{x}, \quad x \neq 0; \qquad h_2(y) = \frac{f_2(y)}{y}, \quad y \neq 0. \tag{4.5}$$

Theorem 4.2. *If* $b h_2(y) > 0$ *for* $y \neq 0$ *and the Conditions* (4.5) *are satisfied, then the trivial solution of system* (4.4) *is asymptotically stable in the large.*

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Behavior of Solutions Near Integral Manifolds

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I. Introduction

Much is known about the behavior of the solutions of the differential system

$$\frac{dx}{dt} = Ax + f(t, x), \quad x = (x_1, \ldots, x_n), \quad f(t, 0) = 0,$$

in a neighborhood of $x = 0$ provided that the eigenvalues of the constant matrix A have nonzero real parts and $f(t, x)$ is small compared with x when x is small (see, *e.g.* [*5*, Ch. 13]). However, if some of the eigenvalues of A have zero real parts, then the theory is not so well developed, although some great contributions have been made by A. LIAPUNOV [*14*] and more recently by DYHMAN [*8*], MALKIN [*16*], PERSIDSKII [*18*] (see also [*11*, Ch. 6]). The purpose of the present paper is to discuss a particular class of equations of the above type (and even more general ones) where some of the roots of A have zero real parts. The relationship of this system to the ones considered by the above mentioned authors is given in Section III. The system of equations considered arises in a natural way as variational equations for invariant surfaces which consist of families of periodic solutions.

As a motivation for what follows, consider the system of autonomous differential equations

$$\frac{dx}{dt} = X(x) \tag{1.1}$$

where x, X are n-vectors, $X \in C^2$ and suppose that $x = x^0(t)$ is a periodic solution of this system. It is known [*5*, p. 323] that this periodic solution is asymptotically orbitally stable (*i.e.*, the closed curve C_0 in x-space defined by $x = x^0(t)$ is asymptotically stable) if $n - 1$ of the characteristic exponents of the linear variational equation

$$\frac{dy}{dt} = \frac{\partial X(x^0)}{\partial x} y \tag{1.2}$$

have negative real parts (necessarily one characteristic exponent is zero). If $x^0(t)$ is a solution of (1.1), then $x^0(t + \varphi)$, where φ is an arbitrary constant, is also a solution and, therefore, the function $x^0(t+\varphi)$ defines a cylinder, C, in the (x,t)-space. If the closed curve C_0 in the x-space is stable, then clearly the cylinder C in the (x,t)-space is stable. But, by discussing the solutions in the (x,t)-space more information can be obtained. In fact, under the same hypotheses on the linear variational equation (1.2), not only is the cylinder asymptotically stable, but for any solution $x(t)$ of (1.1) with $x(0)$ sufficiently close to C_0, there exists a φ_0 such that $x(t)-x^0(t+\varphi_0)\to 0$ as $t\to\infty$ (see, *e.g.*, [*5*, p. 323]). This is called asymptotic orbital stability with asymptotic phase.

Now, suppose that (1.1) has a $(k+1)$-parameter family of periodic solutions $x=x^0[\omega(b)t+\varphi,b]$, where $b=(b_1,\dots,b_k)$ and φ are real constants. In the (x,t)-space, such a family of solutions defines a k-parameter family of cylinders C_b or a $(k+2)$-dimensional manifold $\mathfrak{M}$ of solutions. For a fixed value of b, the variational equation (1.2) can be defined and, in general, has $k+1$ characteristic exponents equal to zero. Is it possible to obtain some type of stability for this manifold $\mathfrak{M}$ if $n-(k+1)$ characteristic exponents of (1.2) have negative real parts for every value of the vector b? Under very weak conditions on $\mathfrak{M}$ this question is answered in the affirmative in Theorem 5.1. In fact, it is shown that for any solution $x(t)$ of (1.1) with $x(0)$ sufficiently close to $\mathfrak{M}$, there exist φ_0, b_0 such that $x(t)-x^0[\omega(b_0)\,t+\varphi_0,b_0]\to 0$ as $t\to\infty$, *i.e.*, $x(t)$ approaches the cylinder C_{b_0} with asymptotic phase.

Another question that naturally arises is the following: suppose that system (1.1) has an asymptotically orbitally stable periodic solution $x^0(t)$ and the right-hand side of (1.1) is perturbed by a function $X^*(t,x)$ which approaches zero as $t\to\infty$ uniformly for all $t\geqq T$ and $x\in U$, a bounded set in x-space. Is it true that the solutions of the perturbed system approach the cylinder defined by $x^0(t+\varphi)$ even though $x^0(t)$ will not, in general, satisfy the perturbed equation? Asymptotically autonomous systems have been previously discussed by L. MARKUS [*17*, p. 27], and he has answered the particular question posed here in the affirmative provided the vector x has dimension two. The technique used by MARKUS seems to depend very strongly on the fact that the phase space is two-dimensional. This question is further investigated in this paper for vectors of arbitrary dimension and $(k+1)$-parameter families of periodic solutions of the unperturbed equation (see Theorems V.6, V.7 and V.15). As can be seen from these theorems, the rapidity with which $X^*(t,x)\to 0$ determines the behavior of the solutions in the neighborhood of the manifold.

All of the results mentioned above are obtained by first introducing local coordinates (s,a,h), where s is a scalar, a is a k-vector, h is an $n-(k+1)$ vector, in a neighborhood of the manifold of solutions of (1.1) in such a way that the manifold is given by $h=0$, $a=$constant, $s=\omega(a)\,t+\varphi$. The differential equations in the new variables s, a, h have the general form given by

$$\begin{aligned}
&\frac{ds}{dt}=\omega(a)+S_1(s,a,h)+S_2(s,a,h,t),\\
(1.3)\qquad &\frac{da}{dt}=A_1(s,a,h)+A_2(s,a,h,t),\\
&\frac{dh}{dt}=H(s,a)\,h+H_1(s,a,h)+H_2(s,a,h,t),
\end{aligned}$$

where each of the functions S_1, A_1, H_1 is $O(\|h\|^2)$ as $\|h\| \to 0$ and the functions S_2, A_2, H_2 behave essentially the same as the perturbing function $X^*(t, x)$ mentioned above. In general, the most that one can hope to obtain is that a solution of (1.3) has the property that $h \to 0$, $a \to$ a constant, c_a, $s - [\omega(c_a)t + \varphi] \to 0$, as $t \to \infty$, where φ is a constant.

The discussion of the behavior of the solutions of a system like (1.3) is difficult since each of the functions $s(t)$, $a(t)$, $h(t)$ is expected to behave in an entirely different manner as a function of t. Although it may be possible to discuss all of the solutions of (1.3) at the same time, we have chosen to look at the equations separately. For example, consider the equation

$$\frac{dh}{dt} = H\big(s(t), a(t)\big)\, h + H_1\big(s(t), a(t), h, t\big) + H_2\big(s(t), a(t), h, t\big) \tag{1.4}$$

where $s(t)$, $a(t)$ are given functions of t. For given functions H, H_1, H_2, how large can the classes of functions $\mathscr{S}$, $\mathscr{A}$ be so that for every $s \in \mathscr{S}$, $a \in \mathscr{A}$, the solution $h(t)$ of (1.4) approaches zero as $t \to \infty$? A result of this type is contained in Lemma II.9. This seems to be a more natural way to attack the problem since the results so obtained may be useful in other applications than the ones considered in this paper (see, *e.g.*, Remark II.11). The other two equations in (1.3) are discussed in the same way, and all of Section II is devoted to a statement of these results. In the proofs of the results of Section II, the fixed point theorem of TYCHONOV is used systematically.

The results of Section II are then used to obtain theorems about the behavior of the solutions of the complete system (1.3) in somewhat the same manner that the method of elimination is used to solve a system of linear algebraic equations. More precisely, the second equation in (1.3) is solved for $a(h, s)(t)$ as a function of h, s, $h \in \mathscr{H}$, $s \in \mathscr{S}$, where $\mathscr{H}$, $\mathscr{S}$ are given classes of functions. It is then shown that $a \in \mathscr{A}$, where $\mathscr{A}$ is a given class of functions. Then, in the first equation, a is replaced by $a(h, s)(t)$, and h is replaced by $h \in \mathscr{H}$ to obtain a solution $s(h) \in \mathscr{S}$. Finally, in the third equation, s, a are replaced by $s(h) \in \mathscr{S}$, $a\big(h, s(h)\big) \in \mathscr{A}$, respectively, to obtain a function $\bar{h} \in \mathscr{H}$. The three functions $\bar{s} = s(\bar{h})$, $a(\bar{h}, \bar{s})$, $\bar{h}$ yield a solution of (1.3) belonging to $\mathscr{S} \times \mathscr{A} \times \mathscr{H}$. These results are contained in Section III. One reason that the proofs of these results are difficult is the fact that $s \to \infty$ as $t \to \infty$, and the asymptotic behavior of s depends upon the asymptotic behavior of a. If either of the variables s or a is absent in (1.3), then much more elementary arguments can be used (see, for example, J. K. HALE & A. STOKES, A Discussion of Differential Equations on Product Spaces, Proceedings of Symposium on Ordinary Differential Equations, University of Mexico, 1959). The relationship of some of the results in Section III to the general problem of stability in "product spaces" (see S. LEFSCHETZ [*11*, p. 117]) is given.

A key requirement in this approach is that the functions $a(h, s)$ and $s(h)$ be continuous in the topologies of the various function spaces. Lemma II.14 shows that this requirement can be fulfilled. The proof of Lemma II.14 is such that it is clear that this method can be applied to a system of n equations, for any n.

In Section IV a transformation, valid in a neighborhood of a family of periodic solutions of (1.1), is given which transforms systems of type (1.1) or perturbations of (1.1) into special cases of systems of differential equations considered in Sections II and III. In Section V, some theorems are given concerning the application of the theorems of Section III to the stability of surfaces of periodic solutions of (1.1) or perturbations of (1.1). Numerous examples are given to illustrate the theorems of Section V.

The applications in Section V have been the motivation for writing this paper. The systems considered in Section V together with the transformation in Section IV lead directly to systems of the type considered in Section III. The theorems in Section III may be read independently of the rest of the paper, but the proofs depend very strongly upon the results in Section II.

II. Notations and basic lemmas

Let R^n be n-dimensional Euclidean space, and let E^n_T be the space of all continuous functions $x(t)$ from $[T, \infty) \to R^n$, $T \geqq 0$. The topology on E^n_T is the compact open topology, *i.e.*, a subbase of neighborhoods in E^n_T is the collection of subsets of E^n_T of the form $M(L, W) = \{f \in E^n_T \mid f(L) \subset W\}$ where L is a compact subset of $[T, \infty)$ and W is an open set in R^n. Since R^n is a metric space, convergence in the compact open topology is equivalent to uniform convergence on compact subsets of $[T, \infty)$ (KELLEY [*10*, p. 230]). Also, since R^n is a metric space and $[T, \infty)$ is the sum of a countable family of compact subsets having the property that any other compact subset of $[T, \infty)$ is covered by some finite subfamily, the space E^n_T is a metric space and, as R^n is complete, E^n_T is complete with respect to its metric (ARENS [*1*]). Furthermore, E^n_T is a linear topological space, and to define the topology it is sufficient to give a system of neighborhoods of the identity. Now, consider the family of pseudo-norms $\{p_n(x)\}$, $p_n(x) = \sup_{T \leqq t \leqq T+n} \|x(t)\|$, where $x \in E^n_T$ and $\|y\|$ for $y \in R^n$ is any vector norm. It can be shown that the family of sets $\{\lambda V_n\}$, $\lambda \geqq 0$, $n = 1, 2, \ldots$, where $V_n = \{x \in E^n_T \mid p_n(x) \leqq 1\}$, considered as a subbase for a system of neighborhoods of $x = 0$, generates a topology on E^n_T which is equivalent to uniform convergence on compact subsets of $[T, \infty)$. Finally, the sets V_n are convex, and, summarizing the above, we have *E^n_T is a complete, locally convex, linear, topological space* (for more details, see BOURBAKI [*3*]).

A set, A, in E^n_T is bounded if the image of A under each pseudo-norm, $p_n(x)$, is bounded. Notice that this does not imply that any function $x \in A$ is bounded on $[T, \infty)$. An operator on E^n_T is a continuous function mapping E^n_T into itself. An operator $\mathscr{I}$ is compact (completely continuous) if for every bounded set $A \subset E^n_T$ the closure of its image under $\mathscr{I}$, $\overline{\mathscr{I}(A)}$, is compact. Let

$$\mathscr{I}(x)(t) = \int_T^t F(x(s), s)\, ds$$

where $F: R^n \times [T, \infty) \to R^n$ is continuous. By using ASCOLI's theorem, the diagonalization procedure and the fact that a sequence $\{x_n\}$ converges in the compact open topology if and only if the sequence $\{x_n(t)\}$ converges uniformly on every compact subset of $[T, \infty)$, it is easy to show that $\mathscr{I}$ is a compact operator.

In the following pages, the fixed point theorem of TYCHONOV [21] will be used systematically.

Theorem (TYCHONOV). *Let E be a complete, locally convex linear topological space, and let $\mathscr{I}: E \to E$ be a continuous operator. If A is a closed, convex subset of E, $\mathscr{I}(A) \subset A$ and $\overline{\mathscr{I}(A)}$ is compact, then there exists a fixed point of $\mathscr{I}$ in A.*

Note that if $\mathscr{I}$ is also a compact operator and A is bounded, then $\overline{\mathscr{I}(A)}$ is necessarily compact. The following sets of functions are useful:

(2.1) $\mathscr{H}(g, T) = \{h \in E_T^r \mid \|h(t)\| \leqq g(t), t \geqq T \geqq 0$, where T is a given constant and g is a given function, $g \in E_T^1$, $g(t) \to 0$ as $t \to \infty$.

(2.2) $\mathscr{S}(\varphi, T) = \{s \in E_T^p \mid \|s(t)\| \leqq \varphi(t), t \geqq T \geqq 0$, where T is a given constant and φ is a given function, $\varphi \in E_T^1$.

(2.3) $\mathscr{A}(m, T) = \{a \in E_T^q \mid \|a(t)\| \leqq m(t), t \geqq T \geqq 0$, where T is a given constant and m is a given function, $m \in E_T^1$, m bounded on $[T, \infty)$.

(2.4) $\mathscr{A}'(c, \psi, T) = \{a \in E_T^q \mid \|a(t) - c\| \leqq \psi(t), t \geqq T \geqq 0$, where $T, c \in R^q$ are given constants and $\psi(t)$ is a given function, $\psi \in E_T^1$, ψ bounded on $[T, \infty)$.

(2.5) $\mathscr{S}'(l, \psi, T) = \{s \in E_T^p \mid \|s(t) - l(t)\| \leqq \psi(t)$, $t \geqq T \geqq 0$, where T is a given constant, and l, ψ are given functions, $l \in E_T^p$, $\psi \in E_T^1$.

Notice that if $\psi(t)$ approaches zero as $t \to \infty$, then in (2.4) each member of $\mathscr{A}'$ approaches c as $t \to \infty$, and in (2.5) the difference between each member of $\mathscr{S}'$ and the function $l(t)$ approaches zero as $t \to \infty$.

The sets defined above are closed, convex and bounded in the compact open topology. The fact that the sets are closed and convex is clearly true, and they are bounded since the definition of a bounded set only requires that the image of the set under each pseudo norm, $p_n(x)$, is a bounded set on the real line. This does not require that the functions be bounded on $[T, \infty)$.

Lemma II.1. *Consider the equation*

$$\frac{da}{dt} = A_1(a, h(t), s(t), t) + A_2(a, h(t), s(t), t), \tag{2.6}$$

where A_1, A_2 are continuous on $\Sigma = \{(a, h, s, t), a \in U \subset R^q, h \in V \subset R^r, s \in R^p, t \geqq 0\}$ and U, V are open bounded spheres with centers at the origin of R^q, R^r respectively, and assume that

$$\|A_1(a, h, s, t)\| \leqq \|h\| \cdot L_1(\|a\|) \quad \text{on} \quad \Sigma, \tag{2.7}$$

where $L_1(r)$ is a given continuous, positive, non-decreasing function for $r \geqq 0$.

(2.8) *For $t \geqq 0$, $a \in U$, $h \in V$ and $s \in E_0^p$, assume that $\|A_2(a, h, s, t)\| \leqq K(t)$, where $K \in E_0^1$ is given, $K(t) \to 0$ as $t \to \infty$ and $\int_0^\infty K(t)\, dt < \infty$.*

If $g \in E_0^1$ *is a given non-negative function,* $g(t) <$ *radius of* V, $t \geqq 0$, $g(t) \to 0$ *as* $t \to \infty$, $\int_0^\infty g(t)\,dt < \infty$; *if* $a_0 \in U$ *is given; if* $C < 0$ *is any constant such that* $\|a_0\| + C <$ *radius of* U; *and if* $T_1 \geqq 0$ *is such that* $\int_{T_1}^\infty g(u)\,du < \int_{\|a_0\|+C}^\infty [L_1(u)]^{-1}\,du$, $\int_{T_1}^\infty K(t)\,dt < C$, *then, for every* $T \geqq T_1$, *every* $s \in E_T^p$, $h \in \mathscr{H}(g, T)$, *there exists a function* $m(t)$, $m \in E_T^1$, $m(t)$ *bounded for* $t \geqq T$, *such that a solution* $a(t)$ *of* (2.6) *satisfying* $a(t) = a_0$ *exists for* $t \geqq T$, $a(t) \in U$, $t \geqq T$, *and* $a(t)$ *lies in* $\mathscr{A}(m, T)$. *Also there exists a function* $f(t)$, $f \in E_T^1$, $f(t) \to 0$ *as* $t \to \infty$ *such that*

$$\|a(t_2) - a(t_1)\| \leqq f(t_1), \qquad t_2 \geqq t_1 \geqq T;$$

and hence there exists a constant $c_a \in U$ *such that* $a(t) \to c_a$ *as* $t \to \infty$. *Also, the constant* T_1, *and the functions* $m(t)$, $f(t)$ *depend only upon the functions* $K(t)$, $g(t)$ *and the constants* a_0 *and* C. *Furthermore, the bound on* $m(t)$ *can be made as close to* $\|a_0\| + C$ *as desired by taking* T *sufficiently large.*

Proof: For any $a_0 \in U$, let $C > 0$ as above be any constant such that $\|a_0\| + C$ is less than the radius of U. Let $T_1 \geqq 0$ be such that $D(T_1) = \int_{T_1}^\infty K(t)\,dt < C$ and $\int_{\|a_0\|+C}^\infty [L_1(u)]^{-1}\,du > \int_{T_1}^\infty g(u)\,du$, where the integral of L_1^{-1} may be infinite, in which case the inequality is trivial. The constant T_1 exists by the hypotheses on $K(t)$ and $g(t)$. Consequently, there exists a solution $m(t)$, $m \in E_T^1$, with $m(T) = \|a_0\| + C$, $T \geqq T_1$, to the equation

$$\frac{dm}{dt} = g(t)\, L_1\big(m(t)\big) \tag{2.9}$$

for every $T \geqq T_1$. Clearly $m(t) \geqq 0$, $m(t) \leqq \|a_0\| + C + M(T)$, $t \geqq T$, where $M(T) \to 0$ as $T \to \infty$. Now choose $T \geqq T_1$ such that $\|a_0\| + C + M(T) <$ radius of U, and then $a(t) \in U$ for $t \geqq T$, if $a \in \mathscr{A}(m, T)$.

With this choice of T and $m(t)$ we define $I^q : E_T^q \to E_T^q$ by the relation

$$I^q(a)\,(t) = a_0 + \int_T^t \{A_1[a(u), h(u), s(u), u] + A_2[a(u), h(u), s(u), u]\}\,du. \tag{2.10}$$

From (2.7), (2.8) and (2.9)

$$\begin{aligned} \|I^q(a)\,(t)\| &\leqq \|a_0\| + \int_T^t \|h(u)\|\, L_1\big(\|a(u)\|\big)\,du + \int_T^t K(u)\,du \\ &\leqq \|a_0\| + C + \int_T^t g(u)\, L_1\big(m(u)\big)\,du = m(t) \end{aligned}$$

for every $s \in E_T^p$, $h \in \mathscr{H}(g, T)$, $a \in \mathscr{A}(m, T)$.

Thus, by TYCHONOV'S theorem, there exists a solution, $a(t)$, of (2.6), $a \in \mathscr{A}(m, T)$, and by our choice of T, $a(t) \in U$, for $t \geqq T$.

Since $a(t) = I^q(a)(t)$, it follows that

$$\begin{aligned} \|a(t_2) - a(t_1)\| &\leqq \int_{t_1}^{t_2} \|A_1[a(u), h(u), s(u), u]\|\, du + \\ (2.11) \qquad &+ \int_{t_1}^{t_2} \|A_2[a(u), h(u), s(u), u]\|\, du \leqq L_1(M_1) \int_{t_1}^{\infty} g(u)\, du + \\ &+ \int_{t_1}^{\infty} K(u)\, du \equiv f(t_1) \end{aligned}$$

for $t_2 \geqq t_1 \geqq T$, where M_1 is a bound for $m(t)$. Clearly $f(t) \to 0$ as $t \to \infty$, and the lemma follows.

Corollary II.2. *If, in addition to the hypotheses of Lemma II.1, the function* A_2 *is identically zero, then for every* $\alpha > 0$, *there exists a constant* $\beta_0(\alpha) > 0$ *such that for all* $\beta \leqq \beta_0(\alpha)$ *and* $h \in \mathscr{H}(\beta \alpha e^{-\alpha t}, 0)$, $s \in E_0^p$, *and* $a_0 \in U$, *there exist a constant* C_2 *and a function* $m(t)$, $m \in E_0^1$, $m(t)$ *bounded for* $t \geqq 0$, *depending only on* β, α, *such that a solution* $a(t)$ *of* (2.6) *with* $a(0) = a_0$ *exists for* $t \geqq 0$, $a(t) \in U$, $t \geqq 0$, $a \in \mathscr{A}(m, 0)$ *and* $a(t)$ *satisfies the relation*

$$\|a(t_2) - a(t_1)\| \leqq C_2 \beta e^{-\alpha t_1}, \qquad t_2 \geqq t_1 \geqq 0;$$

and hence there exists a constant $c_a \in U$ *such that* $a(t) \to c_a$ *as* $t \to \infty$, *and* $a(t) \to c_a$ *exponentially. Furthermore, the bound,* $M_1(\beta)$, *on* $m(t)$ *is given by* $\|a_0\| + C_2 \beta$.

Proof: In the proof of Lemma II.1 let $T_1 = T = 0$, $C = 0$, and choose $\beta_0(\alpha)$ so small that $\alpha \beta_0(\alpha) <$ radius of V and for all $\beta \leqq \beta_0(\alpha)$,

$$\int_{\|a_0\|}^{\infty} [L_1(u)]^{1}\, du > \int_0^{\infty} \beta \alpha e^{-\alpha u}\, du = \beta.$$

Then $m(t)$, the solution to (2.7) with $m(0) = \|a_0\|$, is defined and bounded for $t \geqq 0$, and, by further restricting β, the bound on $m(t)$ may be made so close to $\|a_0\|$ that $a(t) \in U$ for $t \geqq 0$. Furthermore, $f(t)$ in (2.11) is equal to $L_1(M_1) \beta e^{-\alpha t}$, and the conclusion follows.

Remark II.3. It is clear from the proof of Lemma II.1 that, for an arbitrary $a \in \mathscr{A}(m, T)$, $h \in \mathscr{H}(g, T)$, $s \in E_T^p$, the function $a_1 = I^q(a)$, where $I^q(a)$ is defined by (2.10), satisfies the property $\|a_1(t_2) - a_1(t_1)\| \leqq f(t_1)$, $t_2 \geqq t_1 \geqq T$, and f is given by (2.11); *i.e.*, $a_1(t)$ approaches a constant c_{a_1} as $t \to \infty$ and $a_1 \in \mathscr{A}'(c_{a_1}, f, T)$, where $\mathscr{A}'$ is defined by (2.4). Furthermore, the function f depends only on T, g, K and a_0. Now consider the larger class of functions $B = \bigcup_{c \in \overline{U}} \mathscr{A}'(c, f, T)$, with the set U given in Lemma II.1, and $\overline{U}$ is the closure of U. If $a \in B$, the $\lim_{t \to \infty} a(t) = c_a$ exists, and the convergence of a function to its limit is uniform with respect to $a \in B$. This uniformity implies that the map from B into R^q, taking a into c_a, is continuous on B, where the topology on B is that induced by the compact open topology of the function space E_T^q. In fact, for a fixed $a \in B$, let $a_n \in B$, $n = 1, 2, \ldots$, be such that $a_n \to a$ in the compact open topology. We wish to show that $c_{a_n} \to c_a$ as $n \to \infty$. For any $\varepsilon > 0$ choose $t_0 \geqq T$ so large that $f(t_0) < \frac{1}{3}\varepsilon$, and N so large that $\|a_n(t_0) - a(t_0)\| < \frac{1}{3}\varepsilon$ for $n \geqq N$. Then, for every $n \geqq N$, we have

$$\|c_{a_n} - c_a\| \leqq \|c_{a_n} - a_n(t_0)\| + \|a_n(t_0) - a(t_0)\| + \|a(t_0) - c_a\|$$
$$< 2f(t_0) + \tfrac{1}{3}\varepsilon < \tfrac{2}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon.$$

Lemma II.4. *Consider the system of equations*

$$\frac{ds}{dt} = S\big(a(t), h(t), t\big) + S_1\big(a(t), h(t), s, t\big) + S_2\big(a(t), h(t), s, t\big), \tag{2.12}$$

where S, S_1, S_2 *are continuous on* $\Sigma = \{(a, h, s, t),\ a \in U \subset R^q,\ h \in V \subset R^r,\ s \in R^p,\ t \geqq 0\}$ *and* U, V *are open bounded spheres with centers at the origin of* R^q *and* R^r, *respectively. Suppose there exists a constant* L *such that*

$$\|S_1(a, h, s, t)\| \leqq L\|h\| \tag{2.13}$$

for all $(a, h, s, t) \in \Sigma$, *and also there exists a function* $K_1(t)$, $K_1(t) \in E_0^1$, $K_1(t) \to 0$ *as* $t \to \infty$, *such that*

$$\|S_2(a, h, s, t)\| \leqq K_1(t) \tag{2.14}$$

for all $a \in U$, $h \in V$, $s \in R^p$, $t \geqq 0$.

If $g \in E_0^1$ *is a given non-negative function,* $g(t) <$ *radius* V, $t \geqq 0$, $g(t) \to 0$ *as* $t \to \infty$, *then for any constant* $T \geqq 0$, $s_0 \in R^p$, $h \in \mathscr{H}(g, T)$, *and any function* $a \in E_T^q$ *with* $a(t) \in U$ *for* $t \geqq T$, *there exists a function* $\varphi \in E_T^1$, *independent of the particular functions* a, h, *such that there exists a solution* $s(t)$ *of* (2.12), *with* $s(T) = s_0$ *which belongs to* $\mathscr{S}(\varphi, T)$.

Proof: Let $I^p : E_T^p \to E_T^p$ be defined by

$$I^p(s)(t) = s_0 + \int_T^t \big[S\big(a(u), h(u), u\big) + S_1\big(a(u), h(u), s(u), u\big) + S_2\big(a(u), h(u), s(u), u\big)\big]\, du. \tag{2.15}$$

Let $S^*(t) = \sup\big(\|S(a, h, t)\|,\ a \in \overline{U},\ h \in \overline{V}\big)$, where $\overline{U}, \overline{V}$ denote the closure of the sets U, V. Let $\varphi(t) = s_0 + \int_T^t [S^*(u) + Lg(u) + K_1(u)]\, du$. It is immediate that for a, h as given, $I^p\big(\mathscr{S}(\varphi, T)\big) \subset \mathscr{S}(\varphi, T)$, and by the Tychonov fixed point theorem the conclusion follows.

Remark II.5. In the proof of the preceding lemma the transformation I^p is defined even if $a(t)$ is a continuous function of s, h, namely $a(t) = a(s, h)(t)$ provided that, for $s \in \mathscr{S}(\varphi, T)$, $h \in \mathscr{H}(g, T)$, we have $a \in E_T^p$, $a(t) \in U$ for $t \geqq T$. The proof above shows that $I^p(s) \in \mathscr{S}(\varphi, T)$ if $s \in \mathscr{S}(\varphi, T)$ even in this more general situation for φ defined as above.

By Tychonov's theorem, for each $h \in \mathscr{H}(g, T)$, there exists a function $\bar{s} = \bar{s}(h) \in \mathscr{S}(\varphi, T)$ such that $\bar{s} = I^p(\bar{s})$, and this in turn implies that $d\bar{s}(t)/dt = S\big(a(h, \bar{s})(t), h(t), t\big) + S_1\big(a(h, \bar{s})(t), h(t), \bar{s}(t), t\big) + S_2\big(a(h, \bar{s})(t), h(t), \bar{s}(t), t\big)$.

Corollary II.6. *In addition to the hypotheses of Lemma II.4, suppose that* $\int_T^\infty K_1(u)\, du < \infty$, $\int_T^\infty g(u)\, du < \infty$, *and also let* f *be a given non-negative function,* $f \in E_T^1$, $f(t) \to 0$ *as* $t \to \infty$ *and* $\int_T^\infty f(u)\, du < \infty$. *Further, assume that there exists a* *constant* L_1 *such that for* $a_1, a_2 \in U$, $h_1, h_2 \in V$, $t \geqq T$,

$$\|S(a_1, h_1, t) - S(a_2, h_2, t)\| \leqq L_1(\|a_1 - a_2\| + \|h_1 - h_2\|). \tag{2.16}$$

Let $F(t)$ *be defined by*

$$F(t) = \int_t^\infty [L_1 f(u) + (L + L_1) g(u) + K_1(u)]\, du. \tag{2.17}$$

For $a \in \mathscr{A}'(c, f, T)$, $h \in \mathscr{H}(g, T)$, $s \in \mathscr{S}(\varphi, T)$ *there exists a constant* $C_0 = C_0(a, h, s)$ *such that if* $l = l(a, h, s) \in E^p_T$ *is defined by* $l(t) = C_0 + \int_T^t S(c, 0, t)\, dt$, $t \geqq T$, *then we have* $I^p(s) \in \mathscr{S}'(l(a, h, s), F, T)$, *where* $I^p(s)$ *is given by* (2.15). *Furthermore,* $C_0(a, h, s)$, $l(a, h, s)$ *are continuous in* a, h, s *for* $a \in \bigcup_{c \in \bar{U}} \mathscr{A}'(c, f, T) \cap \mathscr{A}(m, T)$, $h \in \mathscr{H}(g, T)$, $s \in \mathscr{S}(\varphi, T)$, *where* $m(t)$ *is a given function,* $m(t) \leqq M_1 <$ *radius* U, $t \geqq 0$, M_1 *constant.*

Proof: If $s_0 \in R^p$ is given, if $\bar{l}(a, h)(t) = s_0 + \int_T^\infty [S(a(u), h(u), u) - S(c, 0, u)]\, du + \int_T^t S(c, 0, u)\, du$, and if $s_1(a, h, s)(t) = I^p(s)(t) - \bar{l}(a, h)(t)$, then

$$s_1(a, h, s, t) = \int_t^\infty [S(c, 0, u) - S(a(u), h(u), u)]\, du + \\ + \int_T^t [S_1(a(u), h(u), s(u), u) + S_2(a(u), h(u), s(u), u)]\, du.$$

By (2.16), (2.13) and (2.14), for every $t_2 \geqq t_1 \geqq T$, we have

$$\|s_1(a, h, s)(t_2) - s_1(a, h, s)(t_1)\| \leqq F(t_1),$$

where $F(t)$ is defined in (2.17), and, therefore, $\lim_{t \to \infty} s_1(a, h, s)(t) \equiv d(a, h, s)$ exists uniformly with respect to $a \in \bigcup_{c \in \bar{U}} \mathscr{A}'(c, f, T) \cap \mathscr{A}(m, T)$, $h \in \mathscr{H}(g, T)$, $s \in \mathscr{S}(\varphi, T)$. If $l(a, h, s)(t) = d(a, h, s) + \bar{l}(a, h)(t)$, then the preceding inequality is clearly satisfied for all $t_2 \geqq t_1 \geqq T$ if the function $s_1(a, h, s)(t)$ is replaced by $I^p(s)(t) - l(a, h, s)(t) = s_1(a, h, s)(t) - d(a, h, s)$. Therefore, letting $t_2 \to \infty$, we have $I^p(s) \in \mathscr{S}'(l(a, h, s), F, T)$. Furthermore, it is clear that

$$l(a, h, s)(t) = C_0(a, h, s) + \int_T^t S(c, 0, u)\, du,$$

$$C_0(a, h, s) = d(a, h, s) + s_0 + \int_T^\infty [S(a(u), h(u), u) - S(c, 0, u)]\, du,$$

and the first part of the lemma is proved.

We now show that $l(a, h, s)$, $C_0(a, h, s)$ are continuous in a, h, s for $a \in \bigcup_{c \in \bar{U}} \mathscr{A}'(c, f, T) \cap \mathscr{A}(m, T)$, $h \in \mathscr{H}(g, T)$, $s \in \mathscr{S}(\varphi, T)$. By Remark II.3, the map taking $a \to c_a$, where c_a is the limit of a as $t \to \infty$, is continuous, and it follows that $\int_T^t S(c, 0, u)\, du$ is continuous. Therefore, $C_0(a, h, s)$ will be continuous if $l(a, h, s)$ is continuous. On the other hand, $l(a, h, s)(t) = d(a, h, s) + \bar{l}(a, h)$, where $d(a, h, s) = \lim_{t \to \infty} s_1(a, h, s)(t)$, $s_1(a, h, s)(t) = I^p(s)(t) - \bar{l}(a, h)(t)$, and their limit exists uniformly over the above sets. Consequently, by using a proof similar to that in Remark II.3, one shows that $d(a, h, s)$ is continuous if $s_1(a, h, s)$ is continuous. But since $I^p(s)$ is certainly continuous, it remains only to show that $\bar{l}(a, h) = b(a, h) + \int_T^t S(c, 0, u)\, du, b(a, h) = s_0 + \int_T^\infty [S(a(u), h(u), u) - S(c, 0, u)]\, du,$ is continuous. But we have already remarked that $\int_T^t S(c, 0, u)\, du$ is continuous, and we now show that $b(a, h)$ is continuous. We need to show that $a_n \to a$, $h_n \to h$ as $n \to \infty$ in the compact open topology implies that $b_n = b(a_n, h_n) \to b = b(a, h)$ as $n \to \infty$. Let $c_n = \lim_{t\to\infty} a_n(t)$, and note from Remark II.3 that $c_n \to c$ as $n \to \infty$. Given $\varepsilon > 0$, choose $t_0 > T$ so large that $\int_{t_0}^\infty L_1[f(u) + g(u)] < \frac{1}{4}\varepsilon$, and choose N so large that $\|a_n(t) - a(t)\| + \|h_n(t) - h(t)\| < \varepsilon/[4L_1(t_0 - T)]$, for $T \leqq t \leqq t_0$, $n \geqq N$, $\|c_n - c\| < \varepsilon/[4L_1(t_0 - T)]$ for all $n \geqq N$. From the definition of $b(a, h)$ above and (2.16), we have

$$\begin{aligned} \|b_n - b\| \leqq & \int_T^{t_0} \|S(a_n, h_n, u) - S(a, h, u)\|\, du + \int_T^{t_0} \|S(c_n, 0, u) - S(c, 0, u)\|\, du + \\ & + \int_{t_0}^\infty \|S(a_n, h_n, u) - S(c_n, 0, u)\|\, du + \int_{t_0}^\infty \|S(a, h, u) - S(c, 0, u)\|\, du \\ \leqq & \int_T^{t_0} L_1(\|a_n(u) - a(u)\| + \|h_n(u) - h(u)\|)\, du + \\ & + \int_T^{t_0} L_1\|c_n - c\|\, du + 2L_1 \int_{t_0}^\infty [f(u) + g(u)]\, du < \varepsilon \end{aligned}$$

for all $n \geqq N$, which completes the proof of Corollary II.6.

Corollary II.7. *In addition to the hypotheses of Lemma II.4, suppose that relation* (2.16) *holds,* $S_2 \equiv 0$, $f(t) = C_2 \beta e^{-\alpha t}$, $\alpha > 0$, *and set* $T = 0$. *Then for* $h \in \mathscr{H}(\beta \alpha e^{-\alpha t}, 0)$, $a \in \mathscr{A}'(c, C_2 \beta e^{-\alpha t}, 0)$, $s \in \mathscr{S}(\varphi, 0)$, *the conclusions of Corollary II.6 are valid, with the addition that the function* $F(t) = L' \beta e^{-\alpha t}$, *for some constant* L'.

Proof: The proof is the same as that of Corollary II.6 with $K_1(t) = 0$, and $T = 0$. The conclusions of Corollary II.6 are immediate, and $F(t)$ is obtained by replacing $f(u)$, $g(u)$ and $K_1(u)$ by $C_2 \beta e^{-\alpha u}$, $\beta \alpha e^{-\alpha u}$, and 0 respectively in (2.17).

Remark II.8. Notice in the proof of Corollary II.6 that the condition $f(t)$ be integrable is not needed if the function S is independent of a; *i.e.*, $a \in E^q_T$, $a(t) \in U$, $t \geqq T$, $h \in \mathscr{H}(g, t)$, $\int_T^\infty g(u)\, du < \infty$ and the Lipschitzian condition on S imply the conclusions of the Corollary.

Lemma II.9. *Consider the equation*

$$\frac{dh}{dt} = H(a(t), s(t), t)\, h + H_1(a(t), h, s(t), t) + H_2(a(t), h, s(t), t) \tag{2.18}$$

where H, H_1, H_2 *are continuous on* $\Sigma = \{(a, h, s, t),\ a \in U < R^q,\ h \in V < R^r,\ s \in R^p,\ t \geqq 0\}$, *and* U, V *are open bounded spheres with centers at the origin of* R^q *and* R^r, *respectively. Suppose that the functions* H, H_1, H_2 *satisfy the following properties* *on* Σ:

(2.19) *Given* $c \in U$, $s \in E_0^p$, *let* $W(c, s, t)$ *denote the matrix solution to* $dW/dt = H(c, s(t), t)\, W$, *with* $W(c, s, 0) = I$, *the identity matrix. Let us suppose that for some* $c \in U$, $l \in E_0^p$, $B > 0$, $\sigma > 0$, *we have* $\|W(c, l, t)\, W^{-1}(c, l, u)\| \leqq B e^{-\sigma(t-u)}$, $0 \leqq u \leqq t$.

(2.20) *There is a constant* $L_1 > 0$ *such that* $\|H(a_1, s_1, t) - H(a_2, s_2, t)\| \leqq L_1(\|a_1 - a_2\| + \|s_1 - s_2\|)$ *for all* $a_1, a_2 \in U$, $s_1, s_2 \in R^p$, $t \geqq 0$.

(2.21) *Given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\|H_1(a, h, s, t)\| \leqq \varepsilon \|h\|$ *for all* $a \in U$, $s \in R^p$, $t \geqq 0$ *and* $\|h\| \leqq \delta$.

(2.22) *There is a continuous function* $K_1(t) \in E_0^1$, $K_1(t) \geqq 0$, $t \geqq 0$, $K_1(t) \to 0$ *as* $t \to \infty$ *such that* $\|H_2(a, h, s, t)\| \leqq K_1(t)$ *for all* $a \in U$, $h \in V$, $s \in R^p$, $t \geqq 0$.

If the above conditions are satisfied, then there exist constants $T' \geqq 0$, ψ', η', *such that for every* $T \geqq T'$, $\psi \leqq \psi'$, $\eta \leqq \eta'$, *there is a function* $g(t) \in E_T^1$, $g(t) \to 0$ *as* $t \to \infty$ *such that for every* $a \in \mathscr{A}'(c, \psi, T)$, $c \in U$, $s \in \mathscr{S}'(l, \psi, T)$ *and every* $h_0 \in R^r$ *with* $\|h_0\| \leqq \eta$, *there is a solution* $h(t)$ *of* (2.18), *existing for* $T \leqq t < +\infty$, $h(T) = h_0$, *and* $h \in \mathscr{H}(g, T)$. *Furthermore,* $\int_T^\infty K_1(u)\, du < \infty$ *implies* $\int_T^\infty g(u)\, du < \infty$, $\int_T^\infty \int_t^\infty K_1(u)\, du\, dt < \infty$ *implies* $\int_T^\infty \int_t^\infty g(u)\, du\, dt < \infty$.

Proof: Let $\gamma > 0$, $\varepsilon > 0$ be numbers such that $B(L_1\gamma + \varepsilon) < \sigma$. Then $\alpha = \sigma - B(L_1\gamma + \varepsilon) > 0$. By (2.21), there is a $\delta > 0$ such that $\|H_1(a(t), h, s(t), t)\| \leqq \varepsilon \|h\|$ for all $a(t) \in U$, $s \in E_0^p$, $t \geqq 0$, $\|h\| \leqq \delta$. Furthermore, we may choose δ so small that the sphere $\|h\| \leqq \delta$ is in V. By (2.22), $\|H_2(a(t), h, s(t), t)\| \leqq K_1(t)$ for every $a(t) \in U$, $s \in E_0^p$, $h \in V$, $t \geqq 0$.

Let $\eta' > 0$ be any number, $\eta' < \delta$, $\eta' < \delta/B$. Since $K_1(t) \to 0$ as $t \to \infty$, we may take $T' \geqq 0$ so large that $B\eta' + (B/\alpha) \sup(K_1(t),\ t \geqq T') < \delta$. For a given $c \in U$ in (2.19) choose the constant $\psi' < \frac{1}{2}\gamma$ so small that if $a \in \mathscr{A}'(c, \psi', T)$ for $T \geqq T'$, then $a(t) \in U$ for all $t \geqq T$. In (2.5), for any $\psi \leqq \psi'$ and $l(t)$ in (2.19), define $\mathscr{S}'(l, \psi, T)$. Therefore, for every $a \in \mathscr{A}'(c, \psi, T)$, $s \in \mathscr{S}'(l, \psi, T)$, $t \geqq T \geqq T'$, $\psi \leqq \psi'$ we have $\|H(a(t), s(t), t) - H(c, l(t), t)\| \leqq L_1\gamma$. Now consider the function $g \in E_T^1$, defined by the differential equation

$$\text{(2.23)} \qquad \frac{dg}{dt} = -\alpha g + B K_1(t), \quad g(T) = B\eta, \quad \eta \leqq \eta',$$

i.e., by the integral equation

$$\text{(2.23)}' \qquad g(t) = e^{-\alpha(t-T)} B\eta + \int_T^t e^{-\alpha(t-u)} B K_1(u)\, du, \quad t \geqq T.$$

Obviously $g(t) \geqq 0$, $g(T) = B\eta$ and $g(t) \leqq B\eta + \sup(K_1(t),\ t \geqq T)(B/\alpha) < \delta$. It is also clear that $g(t) \to 0$ as $t \to \infty$. Another integral equation equivalent to (2.23) is

$$\text{(2.23)}'' \quad g(t) = B\eta\, e^{-\sigma(t-T)} + \int_T^t B\, e^{-\sigma(t-u)} [(L_1\gamma + \varepsilon)\, g(u) + K_1(u)]\, du.$$

Now let $\mathscr{H}(g, T)$ be the class of functions in (2.1) defined by the function $g(t)$, $t \geqq T$, above. For every $h_0 \in V$, $\|h_0\| \leqq \eta$, η above, define the transformation $I^r : E^r \to E^r$ as

$$(2.24) \qquad I^r(h)(t) = W(t)\, W^{-1}(T)\, h_0 + \int_T^t W(t)\, W^{-1}(u) \big[\{H(a(u), s(u), u) - \\ - H(c, l(u), u)\}\, h(u) + H_1\big(a(u), h(u), s(u), u\big) + \\ + H_2\big(a(u), h(u), s(u), u\big)\big]\, du,$$

where $W(t) = W(c, l, t)$ is defined in (2.19). For every $h \in \mathscr{H}$, we have

$$(2.25) \quad \|I^r(h)(t)\| \leqq B\eta\, e^{-\sigma(t-T)} + \int_T^t B\, e^{-\sigma(t-u)} [(L_1\gamma + \varepsilon)\, g(u) + K_1(u)]\, du.$$

From (2.23)″ the second member is $g(t)$, and hence $\|I^r(h)(t)\| \leqq g(t)$, $t \geqq T$. Therefore, $I^r(h)(t) \in \mathscr{H}(g, T)$ and by Tychonov's fixed point theorem, there exists a solution $h(t)$ of (2.18) belonging to $\mathscr{H}$. To complete the proof, it is sufficient to recall that $g(t) \to 0$ as $t \to \infty$, and to observe, from (2.23), that

$$(2.26) \qquad \int_T^t g(u)\, du = \alpha^{-1}[g(T) - g(t)] + B\alpha^{-1} \int_T^t K_1(u) du.$$

Remark II.10. Suppose all the conditions of Lemma II.9 are satisfied except that in (2.19) the bound on $W(c, l, t)\, W^{-1}(c, l, u)$, $0 \leqq u \leqq t$, is supposed to exist for any $c \in U$, $l \in J \subset E_0^p$ for some set J, where B is assumed to be independent of c, l and $\sigma = \sigma(c) > 0$ is independent of l and is a continuous function of c. Given T, ψ and a function $g \in E_T^1$, $g(t) \to 0$ as $t \to \infty$, consider functions $a = a(h, s) \in \mathscr{A}'\big(c(h, s), \psi, T\big) \cap \mathscr{A}(m, T)$, $s = s(h) \in \mathscr{S}'\big(l(h), \psi, T\big) \cap \mathscr{S}(\varphi, T)$, where $a(h, s)$, $c(h, s)$, $s(h)$, $l(h) \in J$ are continuous for $h \in \mathscr{H}(g, T)$, $s \in \mathscr{S}(\varphi, T)$ and $\|c(h, s)\| \leqq d <$ radius of U, for $h \in \mathscr{H}(g, T)$, $s \in \mathscr{S}(\varphi, T)$, where d is a constant. Then there exist constants $T' \geqq 0$, ψ', η' such that for all $T \geqq T'$, $\psi \leqq \psi'$, $\eta \leqq \eta'$, there is a function $g \in E_T^1$, $g(t) \to 0$ as $t \to \infty$ such that for every $h_0 \in R^r$, $\|h_0\| \leqq \eta$ and any functions $a(h, s)$, $c(h, s)$, $s(h)$, and $l(h)$ as above, there is a function $\bar{h} \in \mathscr{H}(g, T)$, $\bar{h}(T) = h_0$ such that $I^r(\bar{h}) = \bar{h}$. This clearly implies

$$\frac{d\bar{h}(t)}{dt} = H\big(a(\bar{h}, s(\bar{h})(t), s(\bar{h})(t), t)\, \bar{h}(t)\big) + H_1\big(a(\bar{h}, s(\bar{h}))(t), \bar{h}(t), s(\bar{h})(t), t\big) + \\ + H_2\big(a(\bar{h}, s(\bar{h}))(t), \bar{h}(t), s(\bar{h})(t), t\big).$$

The proof of this follows from the proof of Lemma II.9, by first letting $\sigma = \min \sigma(c)$, $\|c\| \leqq d$. If α is defined using this value of σ, then the function $g(t)$ in (2.23) is independent of a, s, l, c, $\|c\| \leqq d$. Further, $W(c, l, t)$, as a solution of the equation, $dW/dt = H\big(c(h, s(h)), l(h)(t), t\big)\, W$, is also a continuous function of $h \in \mathscr{H}(g, T)$ (see Lemma II.14 below). Thus the operator I^r is well defined, and the proof asserts that $I^r\big(\mathscr{H}(g, T)\big) \subset \mathscr{H}(g, T)$. The result follows by Tychonov's theorem.

This will be needed in Section 3. This same remark applies to Corollary II.2 and Lemma II.13 below. For simplicity, these lemmas are not stated in this more complicated form.

Remark II.11. Let us analyze some of the implications of the statement and proof of the above lemma. First, if a, s do not appear in (2.18), then Lemma II.9 coincides with a theorem of E. A. CODDINGTON & N. LEVINSON [*5*, p. 327].

Second, in the proof of Lemma II.9, the only restriction on the constant T for the given constants B, η', α, δ, was that $B\eta' + (B/\alpha)\sup\big(K_1(t), t \geqq T'\big) < \delta$. However, if we suppose that the perturbations are of a more restricted character, namely, that the function $K_1(t)$, not necessarily approaching zero as $t \to \infty$, can be made as small as desired for all $t \geqq 0$, then the above inequality can be satisfied for all $t \geqq 0$, and so the function $g(t)$ in (2.23), for $T = 0$, is well defined and $< \delta$ provided $K_1(t)$ is sufficiently small. Then the remainder of the proof of the lemma is exactly the same in order to show that there is a solution $h(t)$ of (2.18), with $h(0) = h_0$, $\|h_0\| \leqq \eta$ and $\|h(t)\| \leqq g(t)$, $t \geqq 0$.

Notice that the equation (2.18) has the solution $h = 0$ for $H_2 \equiv 0$ and the solution $h = 0$ is asymptotically stable. However, for $H_2 \not\equiv 0$, the function $h = 0$ is not generally a solution of (2.18), but with the restricted class of perturbations above, the t-axis in (h, t)-space is stable in the sense of LYAPUNOV. This contains a result of MALKIN [*16*, p. 302]. If we further require that the function $K_1(t) \to 0$ as $t \to \infty$ and that it also can be made as small as desired for $t \geqq 0$, then the t-axis is stable under this type of perturbation, and $h(t) \to 0$ as $t \to \infty$.

Corollary II.12. *In addition to the hypotheses of Lemma II.9 assume that the function H_2 is identically zero. If $\gamma > 0$, $\varepsilon > 0$ are any numbers such that $B(L_1\gamma + \varepsilon) < \sigma$ and $\alpha = \sigma - B(L_1\gamma + \varepsilon)$, then there exist constants ψ' and $\eta' > 0$ such that, for $\psi \leqq \psi'$, $\eta \leqq \eta'$, $s \in \mathscr{S}'(l, \psi, 0)$, $a \in \mathscr{A}'(c, \psi, 0)$, $h_0 \in R^r$, $\|h_0\| \leqq \eta$, there is a solution $h(t)$ of* (2.18), *$h(0) = h_0$, and $h \in \mathscr{H}(B\,\eta\, e^{-\alpha t}, 0)$. Thus $\|h(t)\| \leqq B\,\eta\, e^{-\alpha t}$, $t \geqq 0$.*

Proof: It is clear that T may be chosen equal to zero in this case. Furthermore, in (2.23), $K_1(t)$ may be chosen equal to zero, and the conclusion follows.

Lemma II.13. *Suppose that system* (2.18) *satisfies all of the conditions of Lemma II.9 except that* (2.19) *is replaced by the following stronger hypothesis:*

(2.27) *For $c \in U$, there exists $B > 0$, $\sigma > 0$, so that for every $s \in E_0^p$, we have $\|W(c, s, t)\, W^{-1}(c, s, u)\| \leqq B e^{-\sigma(t-u)}$, $0 \leqq u \leqq t$, where $W(c, s, t)$ is defined in* (2.19).

Then there exist constants $T' \geqq 0$, $\psi' > 0$, $\eta' > 0$ such that for $T \geqq T'$, $\eta \leqq \eta'$, $\psi \leqq \psi'$, there is a function $g \in E_T^1$, $g(t) \to 0$ as $t \to \infty$ such that the conclusions of Lemma II.9 are valid with the set $\mathscr{S}'(l, \psi, T)$ replaced by E_T^p.

Proof: The proof is almost identical with the proof of the preceding lemma except that a new operator $I_1^r\colon E^r \to E^r$ is defined by

$$\begin{aligned} I_1^r(h)(t) = W(t)\, W^{-1}(T)\, h_0 + \int_T^t W(t)\, W^{-1}(u)\, \big[\{H(a(u), s(u), u) - {} \\ - H(c, s(u), u)\}\, h(u) + H_1\big(a(u), h(u), s(u), u\big) + {} \\ + H_2\big(a(u), h(u), s(u), u\big)\big]\, du, \end{aligned} \tag{2.28}$$

where $W(t) = W(c, s, t)$ is defined in (2.19).

The following lemma is basic for the proofs of all of the theorems in Section 3.

Lemma II.14. *Let* $f_1(x, y, z, u, t)$, $f_2(x, y, z, u, t)$ *be continuous on* $R^p \times R^q \times R^r \times R \times R$, *with* f_1 *having values in* R^p, f_2 *in* R^q. *On this domain, assume for* $i = 1, 2$,

$$\|f_i(x_1, y_1, z_1, u, t) - f_i(x_2, y_2, z_2, u, t)\| \leq L_i(\mu, \sigma, \varrho, u, t)\,[\|x_1 - x_2\| + \|y_1 + y_2\| + \|z_1 - z_2\|],$$

where $\mu = \max\|x_j\|$, $\sigma = \max\|y_j\|$, $\varrho = \max\|z_j\|$, $j = 1, 2$, *and* L_i *is continuous in all of its arguments.*

Let $\varphi_1 : R \to R^p$ *be continuous, and define* $I_p : E_0^p \times E_0^q \times E_0^r \to E_0^p$ *by* $I_p(x, y, z)(t) = \varphi_1(t) + \int_0^t f_1(x(u), y(u), z(u), u, t)\,du$. *Let* A_p, A_q, A_r *be bounded subsets of* E_0^p, E_0^q *and* E_0^r, *respectively, and assume that for* $(y, z) \in A_q \times A_r$, *there exists at least one* $x = x(y, z) \in A_p$ *such that* $I_p(x(y, z), y, z) = x(y, z)$.

Let $\varphi_2 : R \to R^q$ *be continuous, and define* $I_p : E_0^q \times E_0^r \to E_0^q$ *by* $I_p(y, z)(t) = \varphi_2(t) + \int_0^t f_2(x(y, z)(u), y(u), z(u), u, t)\,du$.

Assume that for $z \in A_r$ *there exists at least one* $y = y(z) \in A_q$ *such that* $I_q(y(z), z) = y(z)$.

Under these conditions, the function $x(y, z)$ *is well defined (i.e., unique) and continuous on* $A_q \times A_r$, *and* $y(z)$ *is well defined (i.e., unique) and continuous on* A_q.

Remark II.15. The operators I_p and I_q above are defined in this general form so as to include types of integral operators ordinarily encountered in differential equations, that is, those of type I^p or I^q, defined in (2.15) and (2.10) respectively, or the operators I^r or I_1^r, defined in (2.24) and (2.28).

Proof of Lemma II.14: For n a positive integer, let $\|x\|_n = \sup\|x(t)\|$, $0 \leq t \leq n$, and for $y_1, y_2 \in A_q$, $z_1, z_2 \in A_r$, let $x_1 = x(y_1, z_1)$, $x_2 = x(y_2, z_2)$, be any functions such that $I_p(x_j, y_j, z_j) = x_j$, $j = 1, 2$. We shall show that there exists a constant $C(n)$ such that $\|x_1 - x_2\|_n \leq C(n)\,[\|y_1 - y_2\|_n + \|z_1 - z_2\|_n]$. For $y_1 = y_2$, $z_1 = z_2$ it follows that $x_1 = x_2$, *i.e.*, the fixed point is unique. The same indequality clearly implies that this function $x(y, z)$ is continuous.

Let $K_i = \sup L_i(\mu, \sigma, \varrho, u, t)$ for $0 \leq \mu, \sigma, \varrho \leq B$, $u, t \in [0, n]$ where B is the bound over $[0, n]$ on the elements of A_p, A_q, A_r. B exists, as all of these sets are bounded.

Since x_1, x_2 are fixed points of I_p, it follows that

$$\|x_1 - x_2\|(t) \leq K_1 \int_0^t [\|x_1 - x_2\|(u) + \|y_1 - y_2\|(u) + \|z_1 - z_2\|(u)]\,du, \tag{2.29}$$

for all $t \in [0, n]$, where $\|x_1 - x_2\|(t) = \|x_1(t) - x_2(t)\|$, *etc.*

If $r(t) = \int_0^t \|x_1 - x_2\|(u)\,du$, $g(t) = \int_0^t [\|y_1 - y_2\|(u) + \|z_1 - z_2\|(u)]\,du$, then $dr(t)/dt - K_1 r(t) \leq K_1 g(t)$. Since $r(0) = 0$, we have $r(t) \leq \int_0^t e^{K_1(t-u)} K_1 g(u)\,du \leq K_1 e^{K_1 n} \int_0^t g(u)\,du$, or

$$\int_0^t \|x_1 - x_2\|(u)\,du \leq K_1 e^{K_1 n} \int_0^t \int_0^u [\|y_1 - y_2\|(\tau) + \|z_1 - z_2\|(\tau)]\,d\tau\,du. \tag{2.30}$$

From (2.29) and (2.30) it is now clear that the desired inequality is readily obtained, with $C(n) = K_1 n + K_1^2 n^2 e^{K_1 n}$.

Now take $z_1, z_2 \in A_r$, and let $y_j = y(z_j)$, $x_j = x(y_j, z_j)$, be any functions such that $I_q(y_j, z_j) = y_j$, $j = 1, 2$. For n a positive integer, it suffices by the previous remarks to show that there exists a constant $C'(n)$ such that $\|y_1 - y_2\|_n \leqq C'(n)\,\|z_1 - z_2\|_n$.

As before, for $t \in [0, n]$ and using (2.30), we obtain

$$\begin{aligned} \|y_1 - y_2\|(t) &\leqq K_2 \int_0^t [\|x_1 - x_2\|(u) + \|y_1 - y_2\|(u) + \|z_1 - z_2\|(u)]\,du \\ &\leqq K_2 \int_0^t [\|y_1 - y_2\|(u) + \|z_1 - z_2\|(u)]\,du + \\ &\quad + K_1 K_2 e^{K_1 n} \int_0^t \int_0^u [\|y_1 - y_2\|(\tau) + \|z_1 - z_2\|(\tau)]\,d\tau\,du. \end{aligned} \tag{2.31}$$

Now let $p_1(t) = \int_0^t \int_0^u \|y_1 - y_2\|(\tau)\,d\tau\,du$, and put $k(t) = \frac{d^2 p_1(t)}{dt} - K_2 \frac{dp_1(t)}{dt} - K_1 K_2 e^{K_1 n} p_1(t)$, $h(t) = K_2 \int_0^t \|z_1 - z_2\|(u)\,du + K_1 K_2 e^{K_1 n} \int_0^t \int_0^u \|z_1 - z_2\|(\tau)\,d\tau\,du$.

The above inequality is now equivalent to $k(t) \leqq h(t)$ for $t \in [0, n]$. To estimate $p_1(t)$, $dp_1(t)/dt = p_2(t)$, consider the following vector differential equation:

$$\frac{dp}{dt} = A p(t) + \bar{k}(t), \qquad p(0) = 0,$$

where $p = \mathrm{col}(p_1, p_2)$, $A = (a_{ij})$, $a_{11} = 0$, $a_{12} = 1$, $a_{21} = K_1 K_2 e^{K_1 n}$, $a_{22} = K_2$, $\bar{k} = \mathrm{col}(0, k)$. Now,

$$p(t) = \int_0^t e^{A(t-u)} \bar{k}(u)\,du. \tag{2.32}$$

Since $a_{ij} \geqq 0$, it follows that every element of the matrix e^{At} is also non-negative for $t \geqq 0$. Thus, letting $\bar{h} = \mathrm{col}(0, h)$, we see that (2.32) implies

$$p(t) \leqq \int_0^t e^{A(t-u)} \bar{h}(u)\,du, \tag{2.33}$$

since $\bar{k} \leqq \bar{h}$, where the order "$\leqq$" is vector ordering, *i.e.*, $x \leqq y$ if and only if $x_i \leqq y_i$ for all i.

So, if $\bar{A}$ is a bound on every element of e^{At}, for $t \in [0, n]$, (2.33) and (2.31) imply that $\|y_1 - y_2\|(t) \leqq h(t) + \bar{A}(K_1 K_2 e^{K_1 n} + K_2) \int_0^t h(u)\,du$, and since $h(t) \leqq (K_2 n + K_1 K_2 n^2 e^{K_1 n})\,\|z_1 - z_2\|_n$, the desired inequality can be obtained easily.

Remark II.16. Note that if the Lipschitz constants $L_i(\mu, \sigma, \varrho, u, t)$ do not depend upon one of the arguments μ, σ, ϱ, then the corresponding set A_p, A_q, A_r need not be bounded.

Remark II.17. The inversion of the inequality used in the proof of the first part of Lemma II.14 is GRONWALL's inequality, and the proof given here is the standard one (see, *e.g.*, [2, p. 35]). The proof of the second part of Lemma II.14

is a generalization of GRONWALL's inequality. The same result could also be obtained by integrating by parts twice in the right-hand side (2.31) and then applying GRONWALL's inequality. However, the above proof is very easily generalized to the case where more than three integral equations are solved simultaneously.

III. Stability theorems for integral manifolds

In this section we consider systems of equations of the form

$$\begin{aligned} \frac{ds}{dt} &= S(a,h,t) + S_1(a,h,s,t), \\ \frac{da}{dt} &= A_1(a,h,s,t), \\ \frac{dh}{dt} &= H(a,s,t)\,h + H_1(a,h,s,t), \qquad t \geqq 0, \end{aligned} \tag{3.1}$$

and of the form

$$\begin{aligned} \frac{ds}{dt} &= S(a,h,t) + S_1(a,h,s,t) + S_2(a,h,s,t), \\ \frac{da}{dt} &= A_1(a,h,s,t) + A_2(a,h,s,t), \\ \frac{dh}{dt} &= H(a,s,t)\,h + H_1(a,h,s,t) + H_2(a,h,s,t), \qquad t \geqq 0, \end{aligned} \tag{3.2}$$

where $s \in R^p$, $a \in R^q$, $h \in R^r$, and the functions S, S_j, A_j, H, H_j, $j=1, 2$, satisfy certain conditions which will be stated precisely in the theorems below. One condition is that $S_1 = A_1 = H_1 = 0$ for $h=0$, and, therefore, system (3.1) has a family of solutions $h=0$, $a=c$, $ds/dt = S(c, 0, t)$, $s(0) = s_0$, where $c \in R^q$, $s_0 \in R^p$ are arbitrary constants. Theorem III.1 below is concerned with the stability properties of this family of solutions of (3.1). System (3.2) is similar to (3.1) except for the "perturbation" terms S_2, A_2, H_2, which are not necessarily zero for $h=0$. A natural question to ask is the following: What type of "perturbation" terms in (3.2) yield a family of solutions of (3.2) which has essentially the same characteristics as the family mentioned above for the unperturbed equation (3.1)? Theorem III.4 gives a partial answer to this question. In case the a vector is absent in (3.1), (3.2), then the properties of the systems are somewhat different, and this is discussed in Theorems III.7 and III.9.

As mentioned in Section I, applications of these results are given in Section V. The proofs of the following results will be given after Theorem III.9.

Theorem III.1. *Consider the equations*

$$\begin{aligned} \frac{ds}{dt} &= S(a,h,t) + S_1(a,h,s,t), \\ \frac{da}{dt} &= A_1(a,h,s,t), \\ \frac{dh}{dt} &= H(a,s,t)\,h + H_1(a,h,s,t), \end{aligned} \tag{3.3}$$

where s is a p-vector, a is a q-vector, and h is an r-vector. Let $U \subset R^q$, and $V \subset R^r$ be bounded open spheres with center at the origin, and assume that the functions in

(3.3) *satisfy the following properties on* $\Sigma = U \times V \times R^p \times [0, \infty)$:

(3.4) *In* Σ, S, S_1, A_1, H, *and* H_1 *are continuous in* t *and Lipschitzian in* a, h, s *with Lipschitz constant* L;

(3.5) $S_1(a, 0, s, t) = A_1(a, 0, s, t) = 0$ *for* $a \in U$, $s \in R^p$, $t \geqq 0$, *and thus* $\|S_1(a, h, s, t)\| \leqq L\|h\|$, $\|A_1(a, h, s, t)\| \leqq L\|h\|$, *where* L *is the Lipschitz constant given in* (3.4);

(3.6) *Given* $\varepsilon > 0$, *there exists a constant* $\delta > 0$ *such that* $\|H_1(a, h, s, t)\| \leqq \varepsilon \|h\|$, *for* $\|h\| \leqq \delta$, $a \in U$, $s \in R^p$, $t \geqq 0$;

(3.7) *Let* $c \in U$ *be constant, and* $l \in E_0^p$ *be the solution of* $dl/dt = S(c, 0, t)$, $l(0) = l_0$, *for* $t \geqq 0$. *If* $W(t)$ *is the matrix solution of the equation* $dW/dt = H(c, l(t), t)\, W$, $W(0) =$ *the identity matrix, then there exist constants* $B > 0$, $\sigma(c) > 0$ *for all* $c \in U$, $\sigma(c)$ *continuous in* c, *such that* $\|W(t)\, W^{-1}(u)\| \leqq B e^{-\sigma(c)(t-u)}$, *for* $t \geqq u \geqq 0$, *and all* $l_0 \in R^p$.

Under the above assumptions, given $a_1 \in U$, *there exist constants* $\alpha > 0$, $\eta' > 0$ *such that for all* $\eta \leqq \eta'$, $s_0 \in R^p$, $a_0 \in R^q$, $\|a_0\| \leqq \|a_1\|$, *and* $h_0 \in R^r$, $\|h_0\| \leqq \eta$, *the solution* $(s(t), a(t), h(t))$ *of* (3.3) *through* (s_0, a_0, h_0) *at* $t = 0$ *has the following properties:*

(3.8)
$$\|h(t)\| \leqq B \eta e^{-\alpha t}, \quad t \geqq 0;$$

(3.9) $\lim_{t \to \infty} a(t) = c_a \in U$ *exists and* $\|a(t) - c_a\| \leqq c' \eta e^{-\alpha t}$, *for some* $c' > 0$ *and all* $t \geqq 0$;

(3.10) $\|s(t) - l(t)\| \leqq L' \eta e^{-\alpha t}$ *for some* $L' > 0$ *and all* $t \geqq 0$, *where* $l(t)$ *is the solution of* $dl/dt = S(c_a, 0, t)$ *with* $l(0) = C_0$, *for* $t \geqq 0$ *and some* $C_0 = C_0(a, h, s)$.

Remark III.2. A special case of Theorem III.1 has been considered by I. G. Malkin [*16*] where the variable s in (3.3) is *absent* and all of the functions $A_1(a, h, t)$, $H(a, t)$, $H_1(a, h, t)$ are analytic in a, h in a neighborhood of the origin in $R^q \times R^r$. Malkin obtains the same result as above for this special case, namely, in a neighborhood U of the origin in $R^q \times R^r$, every solution $[a(t), h(t)]$ with $[a(0), h(0)] \in U$ has the property that $h(t) \to 0$ exponentially and $a(t) \to c_a$ exponentially as $t \to \infty$.

Remark III.3. Theorem III.1 as well as the theorem of Malkin mentioned above has an interesting relationship to some results of K. P. Persidski [*18*] (see also S. Lefschetz [*11*, p. 117]).

Consider the system of equations

(3.11)
$$\begin{aligned} &\text{a)} \quad dy/dt = Y(y, z, t), \\ &\text{b)} \quad dz/dt = Z(y, z, t), \end{aligned}$$

where $y \in R^q$, $z \in R^r$, $Y(0, 0, t) = Z(0, 0, t) = 0$ *for* $t \geqq 0$, and Y, Z are continuous and Lipschitzian in some neighborhood U of $(0, 0) \in R^q \times R^r$. Let $\zeta(t)$ be a continuous r-vector, and consider the equation

(3.12)
$$dy/dt = Y(y, \zeta(t), t).$$

According to PERSIDSKI, $y=0$ is quasi-stable [quasi-unstable] for (3.11a) whenever the following condition is [is not] fulfilled: Given $\varepsilon>0$ and $t_0 \geqq 0$, there exists an $\eta(\varepsilon, t_0)$, $0<\eta \leqq \varepsilon$ such that if $\zeta(t)$ is any function such that $\|\zeta(t_0)\|<\eta$ and $\|\zeta(t)\| \leqq \varepsilon$, $t \geqq t_0$, then any solution $y(t)$ of (3.12) with $\|y(t_0)\| \leqq \eta$ satisfies $\|y(t)\|<\varepsilon$, $t \geqq t_0$. By symmetry, the same notions are defined for (3.11b). PERSIDSKI then states that if the origin is quasi-stable for both (3.11a) and (3.11b), then the solution $y=0$, $z=0$ of (3.11) is stable in the sense of LYAPUNOV. Also, if the origin is quasi-unstable for one of the two partial systems (3.11a), (3.11b), then the solution $y=0$, $z=0$ of (3.11) is unstable. The last part of this statement is false as the following example shows.

Consider the equation

$$da/dt = a^2 h, \quad dh/dt = -h, \tag{3.13}$$

where a, h are scalars. This system can be easily integrated, or one can observe that it is a special case of the system (3.3) without the s variable and, therefore, there exists a neighborhood U of $(0, 0)$ such that the solution $[a(t), h(t)]$ of (3.13) with $[a(0), h(0)] \in U$ implies $h(t) \to 0$, $a(t) \to a$ constant as $t \to \infty$, and the constant may be made as close to $a(0)$ as desired by taking $h(0)$ sufficiently small. Therefore, the solution $a=0$, $h=0$ of (3.13) is stable in the sense of LYAPUNOV. But the solution $a=0$ of the equation $da/dt = a^2 h$ is quasi-unstable since there are solutions of $da/dt = a^2 \eta$ approaching ∞ in a finite t range for every $\eta>0$.

This example points out how difficult it is to determine conditions for the instability of a system of equations by examining a partial system. For some results along this line, see the paper of M. R. DYHMAN [*8a*].

Theorem III.4. *Consider the equations*

$$\begin{aligned} \frac{ds}{dt} &= S(a, h, t) + S_1(a, h, s, t) + S_2(a, h, s, t), \\ \frac{da}{dt} &= A_1(a, h, s, t) + A_2(a, h, s, t), \\ \frac{dh}{dt} &= H(a, s, t)\, h + H_1(a, h, s, t) + H_2(a, h, s, t), \end{aligned} \tag{3.14}$$

where S, S_1, A_1, and H_1 are as in Theorem III.1, and suppose (3.7) is valid for all $l \in E_0^p$, $c \in U$ *and B, $\sigma(c)$ are independent of $l \in E^p$, with σ continuous on U. If $\Sigma = U \times V \times R^p \times [0, \infty)$, U, V as in Theorem III.1, then assume that S_2, A_2, H_2 are continuous on Σ and Lipschitzian in (a, h, s), and there exists a continuous function $K_1(t) \to 0$ as $t \to \infty$ such that, on Σ,*

$$\|S_2(a, h, s, t)\| \leqq K_1(t), \quad \|A_2(a, h, s, t)\| \leqq K_1(t), \quad \|H_2(a, h, s, t)\| \leqq K_1(t). \tag{3.15}$$

If, in addition $\int_0^\infty K_1(t)\, dt < \infty$, then given $a_1 \in U$, there exists $\eta' > 0$ and $T' \geqq 0$, such that if $T \geqq T'$, $\eta \leqq \eta'$ and $(s(t), a(t), h(t))$ is the solution to (3.14) through (s_0, a_0, h_0) at $t = T$ where $s_0 \in R^p$, $a_0 \in R^q$, $\|a_0\| \leqq \|a_1\|$, $h_0 \in R^r$, $\|h_0\| \leqq \eta$, then $s(t)$ is defined for $t \geqq T$, $\lim_{t\to\infty} a(t) = c_a \in U$ exists and $h(t) \to 0$ as $t \to \infty$.

Moreover, if $\int_0^\infty \int_t^\infty K_1(u)\,du\,dt < \infty$, *then given* $a_1 \in U$, *there exist constants* $\eta' > 0$, $T' \geqq 0$ *such that if* $T \geqq T'$, $\eta \leqq \eta'$, *and* $(s(t), a(t), h(t))$ *is the solution to* (3.14) *through* (s_0, a_0, h_0) *at* $t = T$, *where* $s_0 \in R^p$, $a_0 \in R^q$, $\|a_0\| \leqq \|a_1\|$, $h_0 \in R^r$, $\|h_0\| \leqq \eta$, *then* $h(t) \to 0$, $\lim_{t\to\infty} a(t) = c_a \in U$ *exists, and* $\|s(t) - l(t)\| \to 0$ *as* $t \to \infty$, *where* $dl/dt = S(c_a, 0, t)$, $l(T) = C_0$, $t \geqq T$, $C_0 = C_0(a, h, s)$.

Remark III.5. It will be clear from the proof of this theorem and Remark II.8 that $\|s(t) - l(t)\| \to 0$ as $t \to \infty$, with s, l as above, if $S(a, h, t)$ does not depend on a, provided $\int_0^\infty K_1(t)\,dt < \infty$, even though $\int_0^\infty \int_t^\infty K_1(u)\,du\,dt$ may not be finite. Also, the last conclusion of Theorem III.4 is valid if we require that condition (3.7) is true only for those $l(t)$ for which $dl/dt = S(c, 0, t)$, $c \in U$.

Remark III.6. Without the variable s in (3.14), some closely related results have been obtained by M. R. Dyhman [*8b*].

Theorem III.7. *Consider the equations*

$$\begin{aligned} \frac{ds}{dt} &= S(h,t) + S_1(h,s,t) + S_2(h,s,t), \\ \frac{dh}{dt} &= H(s,t)\,h + H_1(h,s,t) + H_2(h,s,t), \end{aligned} \tag{3.16}$$

where S, S_1, S_2, H, H_1, *and* H_2 *satisfy the conditions of Theorem III.4 with the variable* a *absent.*

Then there exists a $T' \geqq 0$ *and an* $\eta' > 0$ *such that for every* $T \geqq T'$, $\eta \leqq \eta'$, *the solution* $(s(t), h(t))$ *of* (3.16) *through* (s_0, h_0) *at* $t = T$, *where* $s_0 \in R^p$, $h_0 \in R^r$, $\|h_0\| \leqq \eta$, *is such that* $s(t)$ *is defined for* $t \geqq T$, *and* $h(t) \to 0$ *as* $t \to \infty$.

Remark III.8. This theorem is quite elementary and can be proved very easily by other methods than those used here. The following theorem is of a more difficult nature.

Theorem III.9. *Consider the system of equations* (3.16) *where all of the hypotheses of Theorem III.7 are satisfied except that it is required that the matrix solution* $W(t)$ *of* $dW/dt = H(l(t), t)\,W$, *with* $W(0) = I$, *the identity matrix, satisfies* $\|W(t)\,W^{-1}(u)\| \leqq B e^{-\sigma(t-u)}$, $0 \leqq u \leqq t$, $\sigma > 0$, only *for those* $l(t)$ *for which* $dl/dt = S(0,t)$ *for* $t \geqq T$, *and* B, σ *independent of* l. *If* $\int_0^\infty K_1(t)\,dt < \infty$, *then there exist a* $T' \geqq 0$ *and an* $\eta' > 0$, *such that for every* $T \geqq T'$, $\eta \leqq \eta'$, *the solution* $(s(t), h(t))$ *of* (3.16) *through* (s_0, h_0) *at* $t = T$, *where* $s_0 \in R^p$, $h_0 \in R^r$, $\|h_0\| \leqq \eta$, *is such that* $h(t) \to 0$ *as* $t \to \infty$, *and* $\|s(t) - l(t)\| \to 0$ *as* $t \to \infty$, *where* $l(t)$ *is a solution of* $dl/dt = S(0, t)$, $l(T) = C_0$, $t \geqq T$, $C_0 = C_0(s, h)$.

Proof of Theorem III.1: Let $\sigma = \inf \sigma(c)$, $\|c\| \leqq d$, for a fixed d, $\|a_1\| < d <$ radius of U. Using this value of σ, let $\alpha = \sigma - B(L\gamma + \varepsilon)$ as in Corollary II.12.

(i) Consider the differential equation

$$\frac{da}{dt} = A_1[a, h(t), s(t), t], \quad a(0) = a_0, \tag{3.17}$$

where $a_0 \in U$ is a given constant, $\|a_0\| \leqq \|a_1\|$, $s(t)$ is an arbitrary function in E_0^p, $h(t)$ is an arbitrary function in $\mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$, $\beta > 0$, where $\mathscr{H}$ is defined in (2.1) and α given above. The conditions of Corollary II.2 are clearly satisfied, and thus there exists a constant $\beta_0 > 0$ so that for any $\beta \leqq \beta_0$, there exist a constant C_2 (in this case, $C_2 = L$) and a non-negative function $m \in E_0^1$, $m(t) \leqq M_1(\beta)$, $t \geqq 0$, $M_1(\beta) = \|a_0\| + C_2\beta <$ radius of U, such that a solution $a(t) = a(h, s)(t)$ of (3.17), $a(0) = a_0$, satisfies the properties $a(h, s) \in \mathscr{A}(m, 0)$, $\|a(h, s)(t_2) - a(h, s)(t_1)\| \leqq C_2\beta e^{-\alpha t_1}$, $t_2 \geqq t_1 \geqq 0$ for all $s \in E_0^p$, $h \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$. That is, there exists $c(h, s) \in U$, $\|c(h, s)\| \leqq M_1(\beta)$, such that $a(t) \to c(h, s)$ as $t \to \infty$, $a \in \mathscr{A}'(c(h, s), C_2\beta e^{-\alpha t}, 0)$, for $h \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$, $s \in E_0^p$. Furthermore, from Lemma II.14 and Remark II.16, $a(h, s)(t)$ is a continuous function of (h, s) for $h \in \mathscr{H}$, $s \in E_0^p$, and so, by Remark II.3, $c(h, s)$ is also a continuous function of (h, s) on these sets.

(ii) Now, by Remark II.5, there exists a function $\varphi \in E_0^1$ such that for any $h \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$, $\beta \leqq \beta_0$, there is an $s = s(h) \in \mathscr{S}(\varphi, 0)$ such that for $t \geqq 0$,

$$\frac{ds(t)}{dt} = S[a(h, s)(t), h(t), t] + S_1[a(h, s)(t), h(t), s(t), t], \qquad s(0) = s_0, \tag{3.18}$$

where $a(h, s)(t)$ is the function determined above satisfying (3.17). Furthermore, since S is Lipschitzian in a and h, it follows from Corollary II.7 that there exist a $C_0 = C_0(a(h, s(h)), h, s(h)) \equiv C_0(h) \in R^p$ and an $l = l(a(h, s(h)), h, s(h)) \equiv l(h)$, with $dl(h)/dt = S(c(h, s(h)), 0, t))$, $l(h)(0) = C_0(h)$ such that $s(h) \in \mathscr{S}'(l(h), L'\beta e^{-\alpha t}, 0)$, since $s(h) = I^p(s(h))$. Also, from Lemma II.14 and Corollary II.7, the functions $s(h)$, $C_0(h)$ and $l(h)$ are continuous in h, $h \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$.

(iii) Finally, we apply Corollary II.12 and Remark II.10 as follows: In Remark II.10, let J be the set of functions l for which the estimate in (3.7) is valid; that is, $dl/dt = S(c, 0, t)$, $c \in U$, $l(0) \in R^p$ is arbitrary. By (ii) above, clearly $l(h) \in J$, for $h \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$, $\beta \leqq \beta_0$. If we further restrict β_0 so that $M_1(\beta_0) \equiv \|a_0\| + C_2\beta_0 < d$, then from (i), $\|c(h, s(h))\| \leqq d$ for $\beta \leqq \beta_0$ and $h \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$. If ψ', η' are the numbers given in Corollary II.12 and Remark II.10, then we further restrict β_0 in such a way that for $\beta \leqq \beta_0$, we have $C_2\beta \leqq \psi'$, $L\beta \leqq \psi'$, and $\alpha\beta/B \leqq \eta'$. Thus, if ψ, η are defined by $\psi = \max[C_2\beta, L'\beta]$, $\eta = \alpha\beta/B$, then the conclusions of Corollary II.12 and Remark II.10 state that there is a function $\bar{h}(t)$, $\bar{h}(0) = h_0$, $\bar{h} \in \mathscr{H}(\beta\alpha e^{-\alpha t}, 0)$, for $\|h_0\| \leqq \eta = \alpha\beta/B$, and $\bar{h}$ satisfies, for $t \geqq 0$,

$$\frac{d\bar{h}(t)}{dt} = H[a(\bar{h}, s(\bar{h}))(t), s(\bar{h})(t), t]\,\bar{h}(t) + H_1[a(\bar{h}, s(\bar{h}))(t), \bar{h}(t), s(\bar{h})(t), t], \tag{3.19}$$

where $a(h, s(h))$ and $s(h)$ are the functions defined above. The remainder of the proof of the theorem is now obvious since the functions $\bar{s} = s(\bar{h})$, $\bar{a} = a(\bar{h}, \bar{s})$ and $\bar{h}$ are a solution of (3.3) with the desired properties. The Lipschitzian conditions imply the uniqueness of the solution, and the theorem is proved.

Proof of Theorem III.4: Let $T \geqq 0$, $\eta > 0$ be constants to be determined. Let $\sigma = \inf \sigma(c)$, $\|c\| \leqq d$, for a fixed d, $\|a_1\| < d <$ radius of U. Let $\alpha = \sigma - B(L\gamma + \varepsilon)$, where $\gamma > 0$, $\varepsilon > 0$ are any numbers such that $\alpha > 0$. Let $g(t)$ be the solution of (2.23), $dg/dt = -\alpha g + BK_1(t)$, $g(T) = B\eta$. One can show easily, as was done in the proof of Lemma II.9, that $g(t) \to 0$ as $t \to \infty$; that $\int\limits_T^\infty K_1(t)\,dt < \infty$ implies $\int\limits_T^\infty g(t)\,dt < \infty$; and that $\int\limits_T^\infty \int\limits_t^\infty K_1(u)\,du\,dt < \infty$ implies $\int\limits_T^\infty \int\limits_t^\infty g(u)\,du\,dt < \infty$. Also, since $g(t)$ satisfies the above differential equation, it follows as in the proof of Lemma II.9 that $g(t) \leqq B\eta + \sup\bigl(K_1(t), t \geqq T\bigr)(B/\alpha)$, and, thus, there exist $T_0 \geqq 0$, $\eta_0 > 0$, such that for all $T \geqq T_0$, $\eta \leqq \eta_0$, we have $g(t) <$ radius of V for $t \geqq T$. Using $T \geqq T_0$, $\eta \leqq \eta_0$ and $g(t)$ given above, define the class $\mathscr{H}(g, T)$ by (2.1).

(i) First, consider the equation

$$\frac{da}{dt} = A_1[a, h(t), s(t), t] + A_2[a, h(t), s(t), t], \qquad a(T) = a_0, \tag{3.20}$$

where $a_0 \in U$ is a given constant, $\|a_0\| \leqq \|a_1\|$, $s \in E_T^p$ is arbitrary, and h is an arbitrary function belonging to $\mathscr{H}(g, T)$. Since $L_1(r)$ in Lemma II.1 is a constant in the present situation, we have $\int\limits_{\|a_0\|+C}^\infty [L_1(u)]^{-1}\,du = +\infty$ for all C, and the second inequality on g in Lemma II.1 is satisfied. Take $C > 0$, so that $\|a_1\| + C < d$, and further restrict T_0 so that $\int\limits_T^\infty K_1(t)\,dt < C$ for all $T \geqq T_0$. Thus, from Lemma II.1, there exist functions $m, f \in E_T^1$, m bounded for $t \geqq T$, $f(t) \to 0$ as $t \to \infty$, such that a solution $a(t) \equiv a(h, s)(t)$ of (3.20) exists for $t \geqq T$, $a(t) \in U$, $t \geqq T$, $a \in \mathscr{A}(m, T)$ and $\|a(t_2) - a(t_1)\| \leqq f(t_1)$, $t_2 \geqq t_1 \geqq T$. By further restricting T_0, m can be chosen so that $m(t) \leqq d$ for all $t \geqq T \geqq T_0$. Now, by Lemma II.14 and Remark II.16, it follows that $a(h, s)$ is continuous on $\mathscr{H}(g, T) \times E_T^p$. In addition, $a(h, s)(t) \to c(h, s)$ as $t \to \infty$, uniformly on $\mathscr{H}(g, T) \times E_T^p$, since $\|a(h, s)(t) - c(h, s)\| \leqq f(t)$, $t \geqq T \geqq T_0$, and $f(t)$ given in (2.11) is independent of the functions a, h, s over the above sets. Thus, by Remark II.3, $c(h, s)$ is also continuous on $\mathscr{H}(g, T) \times E_T^p$.

(ii) Applying Lemma II.4 and Remark II.5, one can choose a function $\varphi \in E_T^1$ such that for every $h \in \mathscr{H}(g, T)$ there exists a function $s(t) \equiv s(h)(t)$ with $s(T) = s_0$, $s \in \mathscr{S}(\varphi, T)$ and s satisfies, for $t \geqq T \geqq T_0$,

$$\begin{aligned} \frac{ds(t)}{dt} = S[a(h, s)(t), h(t), t] + S_1[a(h, s)(t), h(t), s(t), t] + \\ + S_2[a(h, s)(t), h(t), s(t), t]. \end{aligned} \tag{3.21}$$

Furthermore, φ is independent of the particular function $h \in \mathscr{H}(g, T)$.

(iii) Now, we apply Lemma II.13 and Remark II.10 as follows: In Remark II.10, let $J = E_T^p$. If ψ' is the number given in Lemma II.13 and Remark II.10, then we further restrict T_0 so that $\sup\limits_{t \geqq T_0}\bigl(f(t)\bigr) < \psi'$. Then if $\psi = \sup\limits_{t \geqq T_0}\bigl(f(t)\bigr)$, there exist $\eta' \leqq \eta_0$, $T' \geqq T_0$ such that if g is chosen as indicated in (i) above with $T \geqq T'$, $\eta \leqq \eta'$, $g(T) = B\eta$, then there exists a function $\bar{h} \in \mathscr{H}(g, T)$, $\bar{h}(T) = h_0$, $\|h_0\| \leqq \eta$, and $\bar{h}(t)$ satisfies, for $t \geqq T$,

$$\begin{aligned} \frac{d\bar{h}(t)}{dt} = H\bigl[a\bigl(\bar{h}, s(\bar{h})\bigr)(t), s(\bar{h})(t), t\bigr]\,\bar{h}(t) + H_1\bigl[a\bigl(\bar{h}, s(\bar{h})\bigr)(t), \bar{h}(t), s(\bar{h})(t), t\bigr] + \\ + H_2\bigl[a\bigl(\bar{h}, s(\bar{h})\bigr)(t), \bar{h}(t), s(\bar{h})(t), t\bigr], \end{aligned} \tag{3.22}$$

where $a(h, s(h))$, $s(h)$, are given in (i) and (ii). The remainder of the proof of the first part of Theorem III.4 is now clear.

To prove the second part of the Theorem, assume now that $\int_T^\infty \int_t^\infty K_1(u)\, du\, dt < \infty$. This implies that $\int_T^\infty \int_t^\infty g(u)\, du\, dt < \infty$, and, therefore, $\int_T^\infty f(t)\, dt < \infty$, from the definition of $f(t)$ in equation (2.11). Thus, by Corollary II.6, there exist $C_0(a(h, s(h)), h) \equiv C_0(h)$, $l(a(h, s(h)), h) = l(h)$, with $dl(h)/dt = S(c(h, s(h)), 0, t)$, $l(T) = C_0(h)$, such that $s(h) \in \mathscr{S}'(l(h), F, T)$, where $F \in E_T^1$ is defined by equation (2.17) and $F(t) \to 0$ as $t \to \infty$. Thus, $s(h)(t) - l(h)(t) \to 0$ as $t \to \infty$, and the proof is complete.

Proof of Theorem III.7: The proof of this theorem is almost obvious since it follows so closely the proof of Theorem III.4. Since the variable, a, is absent in (3.16), one need not repeat step (i) in the preceding proof.

Proof of Theorem III.9: This follows by applying the same reasoning as in the proof of Theorem III.4 except omitting (i) and applying Lemma II.4, Corollary II.6 and Remark II.8 in (ii), and Lemma II.9 with Remark II.10 in (iii).

IV. A convenient transformation of variables

Let U_ϱ^p be the p-dimensional solid sphere in R^p with center at the origin and radius ϱ, and let U^p denote an open set in R^p.

Consider the system of differential equations

$$\frac{dx}{dt} = X(x) + X^*(t, x) \tag{4.1}$$

where x, X, X^* are $(n + k + 1)$-dimensional vectors, and suppose that the autonomous system

$$\frac{dx}{dt} = X(x), \tag{4.2}$$

$X \in C^2(U^{n+k+1})$ has a $(k + 1)$-parameter family of periodic solutions

$$\begin{aligned} x &= x^0[\omega(b)\, t + \varphi, b], \\ x^0(s + \pi, b) &= x^0(s, b), \end{aligned} \tag{4.3}$$

where $b = (b_1, \ldots, b_k)$, φ are constants and $\omega(b) > 0$. Naturally $x^0(s, b)$ as a function of s belongs to C^3. We assume that as a function of b it belongs to C^2 and also that $\omega(b) \in C^2(U^k)$*.

Furthermore, we assume that, for all $b \in U^k$, $s \in R$,

$$\operatorname{rank}\left[\frac{\partial x^0(s, b)}{\partial s}, \frac{\partial x^0(s, b)}{\partial b}\right] = k + 1. \tag{4.4}$$

Condition (4.4) is equivalent to the statement that the function $\partial x^0(s, b)/\partial s$, $[\partial x^0(s, b)/\partial s]\,[\partial \omega(b)/\partial b_j]\, t + \partial x^0(s, b)/\partial b_j$, $j = 1, 2, \ldots, k$, are linearly independent, and, furthermore, each of these functions is a solution of the linear variational equations

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* M. Urabe [22] has shown that these conditions are satisfied provided that $\omega(b)$ is bounded in any closed subset of U^k. He also introduced a transformation of coordinates similar to the one discussed in this section.

$$\frac{dy}{dt} = \frac{\partial X[x^0\{\omega(b)\,t+\varphi, b\}]}{\partial x}\,y, \tag{4.5}$$

for every fixed value of b. Therefore, there are $k+1$ chracteristic multipliers of (4.5) equal to one; or, $k+1$ characteristic exponents may be chosen equal to zero. Suppose that the remaining n characteristic multipliers are different from one, and denote the corresponding characteristic exponents by $\alpha_1(b), \ldots, \alpha_n(b)$.

If $s = \omega(b)\,t + \varphi$ in (4.5), then

$$\frac{dy}{ds} = \frac{1}{\omega(b)} \cdot \frac{\partial X[x^0(s,b)]}{\partial x}\,y, \tag{4.6}$$

and if we let

$$p_0(s) = \frac{\partial x^0(s,b)}{\partial s}, \quad p_j(s) = p_0(s)\,\frac{\partial\omega(b)}{\partial b_j}\left[\frac{s-\varphi}{\omega(b)}\right] + \frac{\partial x^0(s,b)}{\partial b_j}, \tag{4.7}$$

$j = 1, 2, \ldots, k$, then it is clear that there is a fundamental system of solutions $Y(s)$ of (4.6) of the form

$$Y(s) = [p_0(s), P(s), Q_1(s,b)\,e^{sH(b)[\omega(b)]^{-1}}] \tag{4.8}$$

where $P(s) = (p_1(s), \ldots, p_k(s))$ is an $(n+k+1)\times k$-dimensional matrix, the functions $p_j(s)$ are given in (4.7), $Q_1(s,b) = Q_1(s+\pi, b)$ is an $(n+k+1)\times n$-dimensional matrix, $Q_1(s,b) \in C^2(R \times U^k)$ and $H(b)$ is an $n \times n$ matrix whose eigenvalues are $\alpha_1(b), \ldots, \alpha_n(b)$. Therefore, every solution of (4.6) can be written in the form $y(s) = p_0(s)\,u + P(s)\,v + Q_1(s,b)\,e^{sH(b)[\omega(b)]^{-1}}\,w_1$ where u, v, w_1 are constants, u is a scalar, v is a k-vector, w_1 is an n-vector. Therefore, the transformation $y \to (u, v, w_1)$ transforms the system of equations (4.6) into an equivalent system $du/ds = 0$, $dv/ds = 0$, $dw_1/ds = 0$. If we let $w_2 = e^{sH(b)[\omega(b)]^{-1}}\,w_1$, then the transformation $(u, v, w_1) \to (u, v, w_2)$ leads to the equivalent system $du/ds = 0$, $dv/ds = 0$, $dw_2/ds = [\omega(b)]^{-1} H(b)\,w_2$. Or, returning to the t variable by the relation $s = \omega(b)\,t + \varphi$, we can summarize this by saying that there exists a matrix $Q(s,b) = Q(s+\pi, b)$ $Q(s,b) = Q_1(s,b)\,e^{H(b)[\omega(b)]^{-1}\varphi}$, such that the transformation of variables

$$y(t) = \frac{\partial x^0(s,b)}{\partial s}\,u + \left[\frac{\partial x^0(s,b)}{\partial s}\,\frac{\partial\omega(b)}{\partial b}\,t + \frac{\partial x^0(s,b)}{\partial b}\right] v + Q(s,b)\,w, \tag{4.9}$$

$s = \omega(b)\,t + \varphi$, transforms system (4.5) into an equivalent system

$$\frac{du}{dt} = 0, \qquad \frac{dv}{dt} = 0, \qquad \frac{dw}{dt} = H(b)\,w. \tag{4.10}$$

Notice that $\det\left[\frac{\partial x^0}{\partial s}, \frac{\partial x^0}{\partial s}\,\frac{\partial\omega(b)}{\partial b}\,t + \frac{\partial x^0}{\partial b}, Q(s,b)\right] \neq 0$ for all $b \in U^k$, where $s = \omega(b)\,t + \varphi$, since all of the above transformations are one-to-one.

The matrices $Q(s,b)$ and $H(b)$ may be complex, but it is known from the POINCARÉ-LYAPUNOV theory (see, *e.g.*, A. M. LYAPUNOV [*14*, p. 399] or I. G. MALKIN [*16*, p. 213 ff.]) that it is always possible to choose both $Q(s,b)$ and $H(b)$ real, $Q(s+\pi, b) = Q(s,b)$, if none of the characteristic multipliers of (4.6) is negative. If some of the characteristic multipliers are negative, then $Q(s, b)$, $H(b)$ may still be chosen to be real matrices provided that it is only required that $Q(s+2\pi, b) = Q(s, b)$, *i.e.*, Q is periodic in s of twice the period of (4.6). These results are summarized in the following lemma.

Lemma IV.1. *If condition* (4.4) *is satisfied, if n of the characteristic multipliers of* (4.5) *are* $\neq 1$, *and if none of the characteristic multipliers of* (4.5) *are negative, then there exist real matrices* $Q(s, b)$, $H(b)$, $Q \in C^2(R \times U^k)$, $H(b) \in C^2(U^k)$, $Q(s+\pi, b) = Q(s, b)$ *of dimensions* $(n+k+1) \times n$ *and* $n \times n$, *respectively, such that the transformation of variables* (4.9) *is one-to-one and transforms system* (4.5) *into an equivalent system* (4.10), *where the eigenvalues of* $H(b)$ *are the nonzero characteristic exponents of* (4.5). *If some of the characteristic multipliers of* (4.5) *are negative, then the matrices* $Q(s, b)$, $H(b)$ *may still be chosen real, provided that* $Q(s+2\pi, b) = Q(s, b)$. *Furthermore,* $Q(s, b)$ *is defined in such a way that* $w = Q(s, b)\, e^{tH(b)}$ *together with the functions in* (4.4) *yield a fundamental system of solutions of* (4.5).

For later reference notice that

$$\frac{\partial x^0(s, b)}{\partial s}\,\omega(b) = X[x^0(s, b)], \tag{4.11}$$

from (4.2) and (4.3), and that

$$\frac{\partial X[x^0(s, b)]}{\partial x}\,Q(s, b) = \frac{\partial Q(s, b)}{\partial s}\,\omega(b) + Q(s, b)\,H(b), \tag{4.12}$$

by differentiating (4.9) and using (4.10), (4.11) and the fact that the functions in (4.4) satisfy (4.5).

Lemma IV.2. *If condition* (4.4) *is satisfied, then there exists a matrix* $Q(s, b)$ *(given in Lemma IV.1) such that the transformation of variables*

$$x = x^0(s, b) + Q(s, b)\,h, \qquad h = (h_1, \ldots, h_n), \tag{4.13}$$

transforms system (4.1) *into an equivalent system,*

$$\begin{aligned} \frac{ds}{dt} &= \omega(b) + \Omega_1(s, b, h) - \Omega_2(s, b, h, t), \\ \frac{db}{dt} &= P_1(s, b, h) + P_2(s, b, h, t), \\ \frac{dh}{dt} &= H(b)\,h + H_1(s, b, h) + H_2(s, b, h, t), \end{aligned} \tag{4.14}$$

for all $h \in U^n_\varrho$ *for some* $\varrho > 0$. *Furthermore, if* $(\partial Q/\partial b)\,h = [(\partial Q/\partial b_1)\,h, \ldots, (\partial Q/\partial b_k)\,h]$,

$$A(s, b, h) = \left[\frac{\partial x^0}{\partial s} + \frac{\partial Q}{\partial s}\,h, \frac{\partial x^0}{\partial b} + \frac{\partial Q}{\partial b}\,h, Q\right]^{-1}, \tag{4.15}$$

then $A(s, b, h) \in C^1(R \times U^k \times U^n)$ *and*

$$\begin{aligned} (\Omega_1, P_1, H_1) &= A(s, b, h)\left[-X(x^0) - \frac{\partial X(x^0)}{\partial x}\,Q h + X(x_0 + Q h)\right], \\ (\Omega_2, P_2, H_2) &= A(s, b, h)\,X^*(t, x^0 + Q h). \end{aligned} \tag{4.16}$$

Proof: Let $Q(s, b)$ be the matrix defined by Lemma IV.1. Then, for a fixed b, $\det[\partial x^0/\partial s, (\partial x^0/\partial s)(\partial\omega/\partial b)\, t + \partial x^0/\partial b, Q(s, b)] \neq 0$, $s = \omega(b)\, t + \varphi$, for all t. But this implies that $\det[\partial x^0/\partial s, \partial x^0/\partial b, Q(s, b)] \neq 0$, for all s. But since each of the elements of this matrix is periodic in s, we have

$$\min_s \det\left[\frac{\partial x^0(s, b)}{\partial s}, \frac{\partial x^0(s, b)}{\partial b}, Q(s, b)\right] \neq 0. \tag{4.17}$$

Now, the transformation $x \to (s, a, h)$ given by (4.13) (notice that when $h = 0$, $s = \omega t + \varphi$, $b =$ constant) yields, from (4.1), (4.11) and (4.12), the equations

$$\begin{aligned}\left(\frac{\partial x^0}{\partial s} + \frac{\partial Q}{\partial s} h\right)\left(\frac{ds}{dt} - \omega\right) + \left(\frac{\partial x^0}{\partial b} + \frac{\partial Q}{\partial b} h\right)\frac{db}{dt} + Q\left(\frac{dh}{dt} - Hh\right)& \\ = - X(x^0) - \frac{\partial X(x^0)}{\partial x} Qh + X(x^0 + Qh) + X^*(t, x^0 + Qh)&.\end{aligned}$$

Since (4.17) is satisfied, there exists a $\varrho > 0$ such that the determinant of the coefficient matrix of the variables $\frac{ds}{dt} - \omega$, $\frac{db}{dt}$, $\frac{dh}{dt} - Hh$ is $\neq 0$ for all $h \in U_\varrho^n$, $b \in U^k$, $s \in R$, and the result follows.

V. Applications to essentially autonomous systems

Consider two systems of differential equations

$$\frac{dx}{dt} = X(x), \qquad t \geqq 0, \tag{5.1}$$

$$\frac{dx}{dt} = X(x) + X^*(t, x), \qquad t \geqq 0, \tag{5.2}$$

where x, X, X^* are real n-vectors and X, X^* satisfy conditions which assure the uniqueness of the solutions of (5.1), (5.2).

Suppose that system (5.1) has a family of solutions $x = x^0[\omega_1(b^1)\, t + \varphi_1, \ldots, \omega_m(b^m)\, t + \varphi_m, b^1, \ldots, b^m]$ where b^j is a vector of dimension k_j, $b^j \in U_j$, where U_j is either an open set in R^{k_j} or consists of a single point, φ_j is a real constant, $-\infty < \varphi_j < +\infty$, and $\omega_j(b^j)$ is a given scalar function, $j = 1, 2, \ldots, m$. Also, suppose there exist real numbers $p_1, \ldots, p_m$ such that $x^0(\vartheta_1, \ldots, \vartheta_j + p_j, \ldots, \vartheta_m, b^1, \ldots, b^m) = x^0(\vartheta_1, \ldots, \vartheta_m, b^1, \ldots, b^m)$ for all $b_j \in U_j$, $-\infty < \vartheta_j < +\infty$, $j = 1, 2, \ldots, m$. This represents a k-parameter family of solutions where $k = \sum_{j=1}^{m} \dim(U_j) + m$.

To understand a little better the nature of such a family of solutions, consider the case $m = 1$. If U_1 is an open set in R^{k_1}, then the solution $x = x^0[\omega(b^1)\, t + \varphi_1, b^1]$ is a $(k_1 + 1)$-parameter family of periodic solutions of (5.1), and if U_1 consists of only a single point, then $x = x^0$ corresponds to a one parameter family of periodic solutions (the only parameter being the arbitrary phase constant φ_1). In case $m = 2$, then $x = x^0$ is a family of doubly periodic solutions and by an appropriate mapping corresponds to a family of tori.

In the following, we shall let $b = (b^1, \ldots, b^m)$ and $U = U_1 \times \cdots \times U_m$. If $b \in U$ is fixed, define C_b as the set of points in (x, t)-space defined by $x = x(\vartheta_1, \ldots, \vartheta_m, b)$, $-\infty < \vartheta_j < +\infty$, $j = 1, 2, \ldots, m$, $-\infty < t < +\infty$, and let $\mathfrak{M} = \bigcup_{b \in U} C_b$. For any subset U^* of U, let $\mathfrak{M}^* = \bigcup_{b \in U^*} C_b$, $N_\sigma(\mathfrak{M}^*)$ be a σ-neighborhood of $\mathfrak{M}^*$

in (x, t)-space and $\mathfrak{N}_\sigma(\mathfrak{M}^*) = [N_\sigma(\mathfrak{M}^*) \cap (R^{n-1} - \mathfrak{M})] \cup \mathfrak{M}^*$. Note that in general $\mathfrak{N}_\sigma(\mathfrak{M}^*)$ is not a neighborhood in the usual sense, but if $\mathfrak{M}^* = \mathfrak{M}$, then $\mathfrak{N}_\sigma(\mathfrak{M}) = N_\sigma(\mathfrak{M})$.

The manifold $\mathfrak{M}$ of solutions of (5.1) will be said to be an *asymptotically stable manifold of* (5.2) if given $\varepsilon > 0$ and $U^* < U$, where U^* is a closed subset of U, there exist constants $\sigma > 0$, $T \geqq 0$ such that every solution $x(t)$ of (5.2) with $(x(T), T) \in \mathfrak{N}_\sigma(\mathfrak{M}^*)$ satisfies the relation $(x(t), t) \in N_\varepsilon(\mathfrak{M}^*)$ for $t \geqq T$ and, in addition $d[(x(t), t), \mathfrak{M}] \to 0$ as $t \to \infty$, where $d(z, A)$ is the distance from a point $z \in R^{n+1}$ to a set $A \in R^{n+1}$. If $\mathfrak{M}$ is an asymptotically stable manifold of (5.2) and for every solution $x(T)$ with $(x(T), T) \in \mathfrak{N}_\sigma(\mathfrak{M}^*)$ there exists a $b_0 \in U$, which depends upon $x(T)$, such that $d[(x(t), t), C_{b_0}] \to 0$ as $t \to \infty$, then we shall say $\mathfrak{M}$ is *asymptotically stable with asymptotic amplitude.* If $\mathfrak{M}$ is an asymptotically stable manifold of (5.2) and for every solution $x(t)$ of (5.2) with $(x(T), T) \in \mathfrak{N}_\sigma(\mathfrak{M}^*)$, there exist $b_0 \in U$, $\varphi_{j0} \in R$, $j = 1, 2, \dots, m$, such that $d[(x(t), t), (y(t), t)] \to 0$ as $t \to \infty$, where $y(t) = x^0[\omega_1(b_0^1)\, t + \varphi_{10}, \dots, \omega_m(b_0^m)\, t + \varphi_{m0}]$, $b_0^1, \dots, b_0^m$, then we shall say $\mathfrak{M}$ is an *asymptotically stable manifold of* (5.2) *with asymptotic amplitude and phase.* In case U consists of a single point, then we shall omit the phrase "asymptotic amplitude".

Notice that the manifold $\mathfrak{M}$ need not be an invariant manifold of (5.2) unless $X^*(t, x)$ is identically zero.

To understand the significance of the above definition, let us compare it with well known examples. First, suppose that $X^*(t, x) = 0$ for all t, x and that $\mathfrak{M}$ consists of a straight line in (x, t)-space, *i.e.*, equation (5.1) has a constant solution x^0. The above definition of stability coincides with asymptotic stability of the solution $x = x^0$ of (5.1) in the sense of Liapunov. As another example, suppose $X^*(t, x) = 0$ for all t, x and that (5.1) has an isolated periodic solution $x = x^0(t)$. Since the system is autonomous, $x = x^0(t + \varphi)$, where φ is an arbitrary constant, is also a solution of (5.1). Therefore, in (x, t)-space, the family of solutions $x = x^0(t + \varphi)$ represents a cylinder $\mathfrak{M}$ of solutions of (5.1). The concept of asymptotic stability of $\mathfrak{M}$ above is equivalent to the asymptotic stability of the orbit of x^0 in the x-space. The concept of asymptotic stability of $\mathfrak{M}$ with asymptotic phase coincides with the results given in E. A. Coddington & N. Levinson [5, p. 323] for this case. The reason for introducing these more general concepts will be clear from the theorems and examples which follow.

Theorem V.1. *Suppose that the system of equations,*

$$\frac{dx}{dt} = X(x), \tag{5.3}$$

where x, X *are real* $(n + k + 1)$*-vectors,* $X \in C^2$, *has a* $(k + 1)$*-parameter family of real periodic solutions,*

$$x = x^0[\omega(b)\, t + \varphi, b], \qquad x^0(s + \pi, b) = x^0(s, b), \tag{5.4}$$

where $b = (b_1, \dots, b_k)$, φ *are real constants,* $x^0(s, b) \in C^3(s)$, $x^0(s, b) \in C^2(b)$, $\omega(b) > 0$, $\omega(b) \in C^2$, $b \in U$, U *is either an open set of* R^k *or a single point**. *If* n *of the char-*

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* If U consists of a single point, then the dependence on b will not be mentioned, and we take $k = 0$. This remark applies to the remainder of the section.

acteristic exponents $\lambda_1(b), \ldots, \lambda_n(b)$, *of the linear variational equation of* (5.4) *have negative real parts for all* $b \in U$, *and*

$$\operatorname{rank}\left[\frac{\partial x^0(s, b)}{\partial s}, \frac{\partial x^0(s, b)}{\partial b}\right] = k + 1 \tag{5.5}$$

for all $b \in U$, $s \in R$, *then the manifold* $\mathfrak{M}$ *defined by* (5.4) *is asymptotically stable with asymptotic phase and amplitude.*

Proof: By Lemmas IV.1 and IV.2 and assumption (5.5), there exists a real matrix $Q(s, b) = Q(s + 2\pi, b)$ (twice the period of x^0) such that in a neighborhood of any periodic solution (5.4), say $b = b_0$, the transformation $x = x^0(s, b) + Q(s, b)\, h$ transforms system (5.3) into an equivalent system

$$\begin{aligned} \frac{ds}{dt} &= \omega(b) + \Omega(s, b, h), \\ \frac{db}{dt} &= P(s, b, h), \\ \frac{dh}{dt} &= H_1(b)\, h + H_1^*(s, b, h), \end{aligned}$$

where the eigenvalues of $H(b)$ are $\lambda_1(b), \ldots, \lambda_n(b)$, provided that $\|b - b_0\|$ is sufficiently small. If we let $b = b_0 + a$, then

$$\begin{aligned} \frac{ds}{dt} &= \omega_1(a) + S_1(a, h, s), \\ \frac{da}{dt} &= A_1(a, h, s), \\ \frac{dh}{dt} &= H(a)\, h + H_1(a, h, s), \end{aligned}$$

and this system is obviously a special case of the system in Theorem III.1, with $S(a, h, t)$ in (3.3) equal to $\omega_1(a)$ and all other functions in (3.3) independent of t.

Remark V.2. From Theorem III.1, it is clear that more has been proved than stated. In fact, not only does $a(t)$ approach a limit c, but also this limit can be made as close to a_0 as desired, and $s(t) - [\omega_1(c)\, t + \varphi'] \to 0$ where φ' is as close to $s(0)$ as desired.

Remark V.3. In case $k = 0$, condition (5.5) of Theorem V.1 is automatically satisfied. The case $k = 0$, *i.e.*, an isolated closed curve in phase space, is exactly the same as Theorem 2.2 of [*5*, p. 323].

Example V.4. Consider the system of equations

$$\begin{aligned} \frac{d^2x}{dt^2} + g(x) &= f(x, y), \\ \frac{dy}{dt} + \gamma(x)\, y &= h(x, y) \end{aligned} \tag{5.6}$$

where x, y are scalars, $g(x), \gamma(x), f(x, y), h(x, y) \in C^2$ for all x, y; $0 = f(x, 0) = h(x, 0) = \partial f(x, 0)/\partial y = \partial h(x, 0)/\partial y$, $g(-x) = -g(x)$, $x g(x) > 0$, $x \neq 0$, $\gamma(x) > 0$, for all x, y. System (5.6) has a two-parameter family of periodic solutions $y = 0$, $x = x^0\big(\omega(b)\, t + \varphi, b\big)$, $x^0(s + \pi, b) = x^0(s, b)$ for all b. In fact, these solutions are defined by the equations

$$\frac{d^2x}{dt^2} + g(x) = 0, \qquad y = 0. \tag{5.7}$$

We assume that $\det(\partial x^0/\partial\varphi, \partial x^0/\partial b) \neq 0$ for all b and $\omega(b) \in C^2$. The linear variational equations for a fixed b are

$$\begin{aligned} &\frac{d^2u}{dt^2} + \frac{\partial g(x^0)}{\partial x} u = 0, \\ &\frac{dv}{dt} + \gamma(x^0)\, v = 0, \end{aligned} \tag{5.8}$$

and from the Floquet theory one of the characteristic exponents of (5.8) is given by

$$\lambda(b) = -\frac{1}{T(b)} \int_0^{T(b)} \gamma[x^0(\omega(b)\,t + \varphi, b)]\, dt$$

where $T(b) = \pi/\omega(b)$. Since $\gamma(x) > 0$ for all x, $\lambda(b) < 0$ for all b. Condition (5.5) of Theorem V.1 is satisfied. Therefore, by Theorem V.1, the manifold of solutions defined by (5.7) is asymptotically stable with asymptotic phase and amplitude.

It is clear that this example could be made much more general. In fact, it is not necessary to suppose $x\,g(x) > 0$, but one need only suppose that system (5.7) has a two-parameter family of periodic solutions.

This example could also be generalized to one of the form

$$\begin{aligned} &\frac{d^2x}{dt^2} + g\left(x, \frac{dx}{dt}\right) = f\left(x, \frac{dx}{dt}, y\right), \\ &\frac{dy}{dt} + \gamma\left(x, \frac{dx}{dt}\right) y = h\left(x, \frac{dx}{dt}, y\right) \end{aligned} \tag{5.9}$$

where $\gamma, g, f, h \in C^2$; f, h and their first derivatives with respect to y vanish for $y = 0$; $\gamma(x, x') > 0$ for all x, x'; and the equation

$$\frac{d^2x}{dt^2} + g\left(x, \frac{dx}{dt}\right) = 0 \tag{5.10}$$

has a two-parameter family of periodic solutions. Conditions under which this last requirement is satisfied have been given (see, *e.g.*, [4]).

Example V.5. Consider the system of equations ($' = d/dt$)

$$x'' + A\,x = \varepsilon f(x, x', \varepsilon) \tag{5.11}$$

where $x = (x_1, \ldots, x_n)$, $A = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, $f \in C^2$, and suppose this system has a two-parameter family of periodic solutions $x = x^0(\omega t + \varphi, b, \varepsilon)$, where φ, b are scalars, $\omega = \omega(b, \varepsilon)$, $\omega(b, 0) = \sigma_1$, $x_1^0(\omega t + \varphi, b, 0) = b \sin(\sigma_1 t + \varphi)$, $x_j^0(\omega t + \varphi, b, 0) = 0$, $j = 2, 3, \ldots, n$. For a fixed b, the linear variational equation is

$$y'' + A\,y = \varepsilon \frac{\partial f(x^0, x^{0\prime}, \varepsilon)}{\partial x} y + \varepsilon \frac{\partial f(x^0, x^{0\prime}, \varepsilon)}{\partial x'} y'. \tag{5.12}$$

This equation has two characteristic exponents equal to zero, and if

$$2\sigma_j \neq 0, \quad \sigma_j \pm \sigma_k \neq 0 \pmod{\sigma_1}, \quad j \neq k, \quad j, k = 2, 3, \ldots, n, \tag{5.13}$$

then it is clear that one can satisfy condition (5.5) for ε sufficiently small. Also, if (5.13) is satisfied, it is known [*9a*] that the real parts of $2n-2$ of the characteristic exponents $\tau_3, \ldots, \tau_{2n}$ of (5.12) are given by

$$R(\tau_{2j-1}) = \frac{\varepsilon}{2T}\int_0^T \frac{\partial f_j}{\partial x_j'}(x^0, x^{0\prime}, 0)\,dt + O(\varepsilon^2), \tag{5.14}$$

$$\tau_{2j} = \bar{\tau}_{2j-1}, \quad j = 2, 3, \ldots, n, \quad T = 2\pi/\sigma_1.$$

Consequently, we have the following result: *If* (5.13) *is satisfied and*

$$\varepsilon\int_0^T [\partial f_j(x^0, x^{0\prime}, 0)/\partial x_j']\,dt < 0, \quad j = 2, 3, \ldots, n, \tag{5.15}$$

then the manifold defined by the two-parameter family of solutions $x = x^0[\omega t + \varphi, b, \varepsilon]$ *of* (5.11) *is asymptotically stable with asymptotic phase and amplitude.*

A particular case in which system (5.11) has such a family of periodic solutions is the following. Let $x_1 = u$, $(x_2, \ldots, x_n) = v$, and then $f(x, x', \varepsilon) = f(u, v, u', v', \varepsilon)$. If $f(u, 0, u', 0, \varepsilon) = 0$, then system (5.11) has a family of solutions $x_1 = b\sin(\sigma_1 t + \varphi)$, $x_k = 0$, $k = 2, 3, \ldots, n$.

Also, one could even assume either that $f_1(u, 0, -u', 0, \varepsilon) = f_1(u, 0, u', 0, \varepsilon)$ or $f_1(-u, 0, u', 0, \varepsilon) = -f_1(u, 0, u', 0, \varepsilon)$ and $f_j(u, 0, u', 0, \varepsilon) = 0$, $j = 2, 3, \ldots, n$ (see, *e.g.*, [*4*]).

An interesting property of the above result is its relationship to a previous result on stability. In case (5.11) has an isolated periodic solution $x(t) = x^0(\omega t + \varphi, \varepsilon)$, $\omega = \omega(\varepsilon)$, $\omega(0) = \sigma_1$, $x_1^0(\omega t + \varphi, 0) = b_0\sin(\sigma_1 t + \varphi)$, $x_k^0(\omega t + \varphi, 0) = 0$, $k = 2, 3, \ldots, n$, then it is known [*9b, 20*] that this periodic solution is asymptotically orbitally stable if $\varepsilon\int_0^T [\partial f_j(x^0, x^{0\prime}, 0)/\partial x_j']\,dt < 0$ for $j = 1, 2, \ldots, n$. The above result is exactly the same as this one except the index j takes only the values $2, 3, \ldots, n$.

Theorem V.6. *Consider the system of equations*

$$\frac{dx}{dt} = X(x) + X^*(t, x), \tag{5.16}$$

where the system $dx/dt = X(x)$ *satisfies the conditions of Theorem V.1, and suppose that there exist continuous functions* $h(y)$, $g(t)$ *and a* $T \geqq 0$ *such that* $g(t) \to 0$ *as* $t \to +\infty$ *and*

$$\|X^*(t, x)\| \leqq g(t)\,h(\|x\|) \tag{5.17}$$

for all x, $t \geqq T$. *If, in addition to these conditions,*

$$\int_T^\infty g(t)\,dt < +\infty, \tag{5.18}$$

then the manifold of solutions (5.4) *of* (5.3) *is an asymptotically stable manifold of* (5.16) *with asymptotic amplitude. Finally, if*

$$\int_T^\infty \int_t^\infty g(u)\,du\,dt < \infty, \tag{5.19}$$

then this asymptotically stable manifold of (5.16) *has asymptotic amplitude and phase.*

Proof: The proof is the same as the proof of Theorem V.1 except that one applies Theorem III.4 instead of Theorem III.1.

In case the manifold of solutions (5.4) is only a one parameter family, *i.e.*, an isolated closed curve in phase space, then the following result holds.

Theorem V.7. *Consider the system of equations* (5.16) *with* $X^*(t, x)$ *satisfying the condition* (5.17). *Then the cylinder in* (x, t)*-space defined by* $x^0(\omega t+\varphi)$ *is an asymptotically stable manifold of* (5.16). *If, in addition,* $g(t)$ *satisfies condition* (5.18), *then this asymptotically stable manifold of* (5.16) *has asymptotic phase.*

Proof: The proof is the same as the proof of Theorem V.1 except that one applies Theorem III.7 instead of III.1.

Let us discuss in more detail the implications of the two previous theorems. Consider first the case where system (5.13) has an isolated asymptotically orbitally stable periodic solution $x^0(\omega t+\varphi)$. In the (x, t)-space this means that the cylinder C of solutions $x^0(\omega t+\varphi)$ is asymptotically stable and, in addition, any solution $x(t)$ of (5.3) with $x(0)$ close enough to the cylinder approaches a particular solution on this cylinder as $t\to\infty$. The first observation to be made about the above two theorems is that the cylinder C defined by the solutions $x^0(\omega t+\varphi)$ of (5.13) need not contain any solutions of (5.16). Secondly, the solutions of (5.16) may not be defined for all $t\geqq 0$, but only for all $t\geqq T$, where T is sufficiently large. Third, the first part of the conclusion of Theorem V.7 states only that there exists a $T\geqq 0$ such that every solution $x(t)$ of (5.16) with $x(T)$ close enough to the cylinder C approaches C as $t\to\infty$, but this solution does not necessarily approach any x^0 as $t\to\infty$. On the other hand, if condition (5.18) is satisfied, then every solution $x(t)$ of (5.16) satisfying the property above approaches a particular one of the functions $x^0(\omega t+\varphi)$ as $t\to\infty$.

Now suppose that system (5.3) has a two-parameter family of periodic solutions, *i.e.*, the solutions generate a one-parameter family of cylinders C_b in the (x, t)-space or a three-dimensional manifold $\mathfrak{M}$ solutions. The first part of Theorem V.6 states that $\mathfrak{M}$ is asymptotically stable with asymptotic amplitude; that is, the solutions "close to" $\mathfrak{M}$ approach one of the cylinders C_b as $t\to\infty$, but do not necessarily approach any x^0 as $t\to\infty$. On the other hand, if (5.19) is satisfied, then the solutions do approach some x^0 as $t\to\infty$. One interesting relationship to notice between Theorems V.6 and V.7 is that the function $X^*(t, x)$ must approach zero faster in Theorem V.6 than in Theorem V.7 to obtain essentially the same qualitative behavior near the manifold of solutions. These remarks are illustrated by the following examples.

Example V.8. Consider the second order equation

$$\frac{d^2x}{dt^2}+\varphi\left(x,\frac{dx}{dt}\right)=\psi\left(t,x,\frac{dx}{dt}\right) \tag{5.20}$$

where $\varphi, \psi\in C^2$ with respect to x, dx/dt; there exists a $T\geqq 0$ such that ψ is continuous in t for $t\geqq T$; and there exist continuous functions $g(t)$, $h(x, y)$ such that

$$|\psi(t,x,y)|<g(t)\,h(\|x\|,\|y\|),\qquad t\geqq T, \tag{5.21}$$

for all x, y and $g(t) \to 0$ as $t \to \infty$. Also, suppose that the autonomous system

$$\frac{d^2 x}{dt^2} + \varphi\left(x, \frac{dx}{dt}\right) = 0 \tag{5.22}$$

has an isolated periodic solution $x^0(\omega t + \varphi)$ whose nonzero characteristic exponent is less than zero. This implies that the cylinder of solutions of (5.22) generated by $x^0(\omega t + \varphi)$ is asymptotically stable with asymptotic phase. Many papers have been written concerning the existence of such a periodic solution of (5.22) (see, *e.g.*, [*4*, chapter 9]). System (5.20) satisfies the conditions of Theorem V.*7*, and, therefore, the cylinder generated by $x^0(\omega t + \varphi)$ in the (x, t)-space is an asymptotically stable manifold of (5.20). If, in addition, $\int_T^\infty g(t)\, dt < +\infty$, then this cylinder is an asymptotically stable manifold of (5.20) with asymptotic phase.

Now consider the special case where $\varphi(x, y) = \mu_0 f(x)\, y + G(x)$, $\mu_0 > 0$, $\psi(t, x, y) = [\mu_0 - \mu(t)]\, f(x)\, y$, where $\mu(t)$ is continuous for $t \geqq 0$, $\mu(t) \to \mu_0$ as $t \to \infty$, $G(-x) = -G(x)$, $f(-x) = f(x)$, $x G(x) < 0$ for $x \neq 0$ and $F(x) = \int_0^x f(t)\, dt$ is negative for $0 < x < \bar{x}$ and positive for $x > \bar{x}$ for some finite $\bar{x}$ and $F(x) \to +\infty$ monotonically as $\|x\| \to \infty$. The equation $x'' + \mu_0 f(x)\, x' + G(x) = 0$ has a unique solution $x^0(t)$ which has a negative characteristic exponent (N. Levinson & O. K. Smith, [13]). Consequently the manifold generated by $x^0(\omega t + \varphi)$ in (x, t)-space is an asymptotically stable manifold of the equation $x'' + \mu(t)\, f(x)\, x' + G(x) = 0$. This result has been previously obtained by L. Markus [*17*, p. 27]. The result of Markus is actually more general than the one here since he shows the solution is globally asymptotically stable. This can be done in two-dimensional phase space, but as we shall see in Example V.*17*, a result similar to the one given here is valid in higher dimensions. Also, we get the further result here that if $\int_T^\infty |\mu(t) - \mu_0|\, dt < +\infty$ for some $T \geqq 0$, then the above manifold is an asymptotically stable manifold with asymptotic phase.

Example V.9. Consider the system of equations

$$\begin{aligned} \frac{d^2 x}{dt^2} + g(x) &= f(x, y) + \varphi(t)\, F(x, y), \\ \frac{dy}{dt} + \gamma(x, y &= h(x, y) + \psi(t)\, H(x, y) \end{aligned} \tag{5.23}$$

where $\gamma, g, f, h, F, H \in C^2$, $\varphi, \psi \in C^1$, $\varphi \to 0$, $\psi \to 0$, as $t \to \infty$ and $\int_0^\infty |\varphi(t)|\, dt < +\infty$, $\int_T^\infty |\psi(t)|\, dt < +\infty$ and suppose that the limiting equation

$$\begin{aligned} \frac{d^2 x}{dt^2} + g(x) &= f(x, y), \\ \frac{dy}{dt} + \gamma(x)\, y &= h(x, y) \end{aligned} \tag{5.24}$$

satisfies the same conditions as system (5.6). Then the asymptotically stable manifold of solutions of (5.24) is an asymptotically stable manifold of (5.23) with asymptotic amplitude. This follows immediately from Theorem V.6. If, in addition $\int_T^\infty \int_t^\infty |\varphi(u)|\,du\,dt < \infty$, $\int_T^\infty \int_t^\infty |\psi(u)|\,du\,dt < \infty$, then, from Theorem V.6, this asymptotically stable manifold has asymptotic amplitude and phase. Obviously, this example could be generalized as in Example V.4.

Theorem V.10. *Consider the system of equations*

$$\begin{aligned} \frac{dx}{dt} &= X(x) + X_1(t, x, y) \\ \frac{dy}{dt} &= P(x)\,y + Y_1(t, x, y) \end{aligned} \tag{5.25}$$

where x, y are vectors; $P, X, X_1, Y_1 \in C^2$ with respect to x, y; and X_1, Y_1 are continuous and bounded in t for all x, y in some bounded region, and $X_1(t, x, 0) = 0$, $Y_1(t, x, 0) = 0$, $\partial Y_1(t, x, 0)/\partial y = 0$. Furthermore, suppose that the system $x' = X(x)$ satisfies all of the properties of system (5.3) of Theorem V.1; in particular, there is a manifold, $\mathfrak{M}$, of solutions of this equation, defined by $x = x^0[\omega(b)\,t + \varphi, b]$, which is asymptotically stable with asymptotic amplitude and phase. If, in addition, the characteristic exponents of the periodic system

$$\frac{dy}{dt} = P[x^0(\omega(b)\,t + \varphi, b)]\,y \tag{5.26}$$

have negative real parts for every b, then the manifold $\mathfrak{M}_1$ defined by $x = [\omega(b)\,t + \varphi, b]$, $y = 0$ is an asymptotically stable manifold of solutions of (5.25) with asymptotic amplitude and phase.

Proof: The functions $x = x^0[\omega(b)\,t + \varphi, b]$, $y = 0$ are a solution of the equations, $dx/dt = X(x)$, $dy/dt = P(x)\,y$, and the linear variational equations are given by

$$\frac{dx}{dt} = \frac{\partial X(x^0)}{\partial x}\,x, \qquad \frac{dy}{dt} = P(x^0)\,y, \tag{5.27}$$

where the coefficients are periodic of period π in the variable $s = \omega(b)\,t + \varphi$. Let the real matrix $Q(s, b) = Q(s + 2\pi, b)$ associated with the first equation in (5.27) be defined by Lemma IV.1. In the same way, let $R(s, b)$ be the real nonsingular matrix associated with the second equation in (5.27) defined by Lemma IV.1.

Observe that

$$\begin{aligned} &\frac{\partial x^0}{\partial s}\,\omega(b) = X[x^0(s, b)], \\ &\frac{\partial X[(x^0(s, b)]}{\partial x}\,Q(s, b) = \frac{\partial Q(s, b)}{\partial s}\,\omega(b) + Q(s, b)\,H(b), \\ &\frac{\partial R}{\partial s}\,\omega(b) + R(s, b)\,L(b) = P[x^0(s, b)]\,R(s, b), \end{aligned} \tag{5.28}$$

where the eigenvalues of the matrices $H(b)$ and $L(b)$ are the nonzero characteristic exponents of the first and second equations (5.27), respectively. These matrices are determined by Lemma IV.1. Now, if we apply the transformation

$x = x^0(s, b) + Q(s, b)\, h$, $y = R(s, b)\, z$ to system (5.25) and use (5.28), we obtain

$$\left(\frac{\partial x^0}{\partial s} + \frac{\partial Q}{\partial s} h\right)\left(\frac{ds}{dt} - \omega\right) + \left(\frac{\partial x^0}{\partial b} + \frac{\partial Q}{\partial b} h\right)\frac{db}{dt} + Q(s, b)\left(\frac{dh}{dt} - H\right)$$

$$= X(x^0 + Qh) - X(x^0) - \frac{\partial X(x^0)}{\partial x} Qh + X_1(t, x^0 + Qh, Rz),$$

$$\frac{\partial R}{\partial s} z\left(\frac{ds}{dt} - \omega\right) + \frac{\partial R}{\partial b} z \frac{db}{dt} + R\left(\frac{dz}{dt} - Lz\right)$$

$$= P(x^0 + Qh)\, Rz - P(x^0)\, Rz + Y_1(t, x^0 + Qh, Rz).$$

Now, by assumption (5.5) concerning the system $dx/dt = X(x)$ and the fact that $R(s, b)$ is nonsingular, the determinant of the coefficient matrix of the variables $ds/dt - \omega(b)$, db/dt, $dh/dt - Hh$, and $dz/dt - Lz$ in the preceding system is different from zero for $h = z = 0$. Therefore, for h, z sufficiently small, one can solve for these variables to obtain

$$\begin{aligned}
&\frac{ds}{dt} - \omega(b) = S_1(b, h, z, s, t),\\
&\frac{db}{dt} = A_1(b, h, z, s, t),\\
(5.29)\qquad &\frac{dh}{dt} - H(b)\, h = H_1(b, h, z, s, t),\\
&\frac{dz}{dt} - L(b)\, z = Z_1(b, h, z, s, t) + R^{-1}\frac{\partial R}{\partial s} z\, S_1(b, h, z, s, t) +\\
&\qquad\qquad + R^{-1}\frac{\partial R}{\partial b} z\, A_1(b, h, z, s, t,)
\end{aligned}$$

where S_1, A_1, H_1 are $O(\|h\|^2 + \|z\|)$ while Z_1 is $O[\|z\|(\|h\| + \|z\|)]$ as $\|h\|$, $\|z\| \to 0$. It is now clear that this is a special case of the system considered in Theorem (3.1) since the functions on the right-hand sides of (5.29) have continuous first derivatives with respect to s, b, h, z and are continuous and bounded in t. The remainder of the proof follows immediately by applying Theorem III.1.

Remark V.11. It is not at all obvious when one can be assured that the characteristic exponents of system (5.26) have negative real parts for every b. One would hope to give conditions for this which depend only upon the matrix $P(x)$ in (5.25). Using a result of A. WINTNER [*23*] (see also R. CONTI [*6*]), one can prove the following result: *If the eigenvalues of the matrix* $B(x) = 2^{-1}[P(x) + P^T(x)]$ (P^T is the transpose of P) *have negative real parts for every x in a neighborhood of the manifold $\mathfrak{M}$, then the characteristic exponents of* (5.26) *have negative real parts.* In fact, if $\Lambda(t, b)$ is the maximal eigenvalue of the matrix $B[x^0(\omega(b)t + \varphi, b)]$, then the result of WINTNER implies that every solution $x(t)$ of (5.26) satisfies the property $\|x(t)\|^2 \leqq \|x_0\|^2 \exp 2\int_0^t \Lambda(\tau, b)\, d\tau$. But $\Lambda(t, b)$ is negative for all t, b and periodic in t, and the result follows. In particular, the above result implies that the characteristic exponents of (5.26) have negative real parts if $P(x)$ is a symmetric matrix whose eigenvalues are negative for all x.

Remark V.12. Theorem V.10 can obviously be generalized along the same lines as Theorems V.6 and V.7 but will not be discussed here since the details can be so easily supplied.

Example V.13. Consider the system $('=d/dt)$

$$(5.30)\qquad \begin{aligned} x''+g(x) &= f(t,x,y), \\ y'+\gamma(x,x')\,y &= h(t,x,y) \end{aligned}$$

where x is a scalar, y is a scalar, $g, \gamma, f, h \in C^2$ with respect to x, y, f, h are continuous and bounded in t for all x, y in some bounded region; $f(t,x,0)=0$, $h(t,x,0)=0$, $\partial h(t,x,0)/\partial y=0$ for all t, x and the equation, $x''+g(x)=0$, has a family of periodic solutions $x=x^0[\omega(b)\,t+\varphi, b]$, where φ, b are scalars, $\det[\partial x^0/\partial\varphi, \partial x^0/\partial b] \neq 0$, and $\omega(b)\in C^2$. If $\gamma(x,x')>0$ for all x, x' [or, if y is a vector and $\gamma(x,x')$ is a matrix which satisfies the properties of Remark V.11], then (5.30) is a special case of system (5.25) if we put $x_1=x$, $x_2=x'$. Therefore, for f, h bounded functions of t, the manifold of solutions defined by $x=x^0[\mathrm{P}(b)\,t+\varphi, b]$, $y=0$ is an asymptotically stable manifold of solutions of (5.30) with asymptotic amplitude and phase. This is a generalization of Example V.4 in that the variable t is introduced in the functions f, h and $\partial f(t,x,0)/\partial y$ may be $\neq 0$.

Example V.14. As an illustration of the case where $\gamma(x,x')$ in (5.30) may be a matrix, consider the system $('=d/dt)$;

$$(5.31)\qquad \begin{aligned} & x''+g(x)=f(x,x',y,y') \\ & y_j''+u_j(x,x')\,y_j'+v_j(x,x')\,y_j=h_j(x,y)\,y_j', \qquad j=1,2,\ldots,n, \end{aligned}$$

where $y=(y_1,\ldots,y_n)$, $f, u_j, v_j, h_j\in C^2$, $f(x,x',0,0)=0$, $h_j(x,0)=0$ for each $j=1,2,\ldots,n$, and suppose that the equation $x''+g(x)=0$ has a family of periodic solutions $x=x^0[\omega(b)\,t+\varphi, b]$, where b, φ are real scalars, $\det[\partial x^0/\partial\varphi, \partial x^0/\partial b]\neq 0$, and $\omega(b)\in C^2$. The manifold $x=x^0$, $y=y'=0$, is a solution of (5.31), and (5.31) will be a special case of system (5.25) of Theorem V.10 if we can determine conditions under which the solutions of the linear equations,

$$(5.32)\qquad y_j''+u_j(x^0,x^{0\prime})\,y_j'+v_j(x^0,x^{0\prime})\,y_j=0, \qquad j=1,2,\ldots,n,$$

approach zero as $t\to\infty$. Since this is a linear equation with periodic coefficients, the approach will necessarily be exponential. Many papers have been devoted to giving conditions which imply that the origin is an asymptotically stable solution of (5.32) (see, *e.g.*, [*4*, p. 63], [*19*, p. 210]). In particular, if

$$(5.33)\quad v_j(x^0,x^{0\prime})>0,\quad 2u_j(x^0,x^{0\prime})\,v_j(x^0,x^{0\prime})+\frac{dv_j}{dt}(x^0,x^{0\prime})>0,\quad j=1,2,\ldots,n,$$

for every b, where $x^0=x^0[\omega(b)\,t+\varphi, b]$, then $y_j=0$, $j=1,2,\ldots,n$ is an asymptotically stable solution of (5.32) [*12*]. Therefore, the manifold $x=x^0$, $y=0$ of solutions of (5.31) is asymptotically stable with asymptotic amplitude and phase if (5.33) is satisfied. Condition (5.33) is certainly satisfied if $v_j=$ constant >0 and $u_j(x,x')>0$, $j=1,2,\ldots,n$, for all x. This last result has been obtained by E. W. THOMPSON [*20*] for the case where $g(x)$ in (5.31) is linear in x.

So far, the theorems of Section 3 have not been used in all their generality; namely, the variable s in the previous applications has always been a scalar. The following two theorems are given to illustrate how situations may arise where s is a vector.

Theorem V.15. *Consider the system of equations*

$$\frac{dx}{dt} = X(x) + X^*(t, x, y), \qquad (5.34)$$
$$\frac{dy}{dt} = Y(y) + Y^*(t, x, y)$$

where x, y *are vectors;* $X, Y, X^*, Y^* \in C^2$*; each of the systems* $dx/dt = X(x)$, $dy/dt = Y(y)$ *has a manifold of solutions defined by* $x = x^0[\omega(b)\,t + \varphi, b]$, $y = y^0[\nu(c)\,t + \varphi, c]$, *respectively, where* φ *is a constant scalar,* b, c *are constant vectors and each of these manifolds satisfies the conditions of Theorem V.1. Also, suppose there exist continuous functions* $g(t)$, $h_1(u, v)$, $h_2(u, v)$ *and a* $T \geqq 0$ *such that* $g(t) \to 0$ *as* $t \to +\infty$ *and*

$$(5.35) \qquad \|X^*(t, x, y)\| \leqq g(t)\, h_1(\|x\|, \|y\|), \qquad \|Y^*(t, x, y)\| \leqq g(t)\, h_2(\|x\|, \|y\|),$$

for all $t \geqq T$ *and* x, y *in some neighborhood of the manifold* $\mathfrak{M}$*, defined by* $x = x^0$, $y = y^0$*. If*

$$(5.36) \qquad \int_T^\infty g(t)\, dt < +\infty,$$

then $\mathfrak{M}$ *is an asymptotically stable manifold of* (5.33) *with asymptotic amplitude. If, in addition,*

$$(5.37) \qquad \int_T^\infty \int_t^\infty g(u)\, du\, dt < \infty,$$

then $\mathfrak{M}$ *also has asymptotic phase.*

In case x^0, y^0 *are isolated periodic solutions, then* $\mathfrak{M}$ *can be mapped onto a torus in* (x, y)*-space. In this case, this torus is an asymptotically stable manifold of* (5.34) *even if* (5.36) *is not satisfied. If* (5.36) *is satisfied, then this torus also has asymptotic phase.*

Proof: From Lemma IV.1, we may define matrices $Q_1(s_1, b)$, $Q_2(s_2, b)$ associated with the variational equations $x' = \{\partial X[x^0(s_1, b)]/\partial x\}\, x$, $y' = \{\partial Y[y^0(s_2, c)]/\partial y\}\, y$, $s_1 = \omega(b)\, t + \varphi$, $s_2 = \nu(c)\, t + \varphi$, respectively. Then the transformation $x = x^0(s_1, b) + Q_1(s_1, b)\, h_1$, $y = y^0(s_2, c) + Q_2(s_2, c)\, h_2$ transforms system (5.34) into an equivalent system

$$\frac{ds}{dt} = \mu(a) + \Omega_1(a, h, s) + \Omega_2(a, h, s, t),$$
$$\frac{da}{dt} = P_1(a, h, s) + P_2(a, h, s, t),$$
$$\frac{dh}{dt} = H(a)\, h + H_1(a, h, s) + H_2(a, h, s, t),$$

where $s = (s_1, s_2)$, $a = (b, c)$, $h = (h_1, h_2)$, $\mu(a) = [\omega(b), \nu(c)]$, $H(a) = \operatorname{diag}\, [H_1^*(b), H_2^*(c)]$, where the eigenvalues of $H_1^*(b)$, $H_2^*(c)$ have negative real parts for every b, c. The functions Ω_1, P_1, H_1 are $O(\|h\|^2)$ as $\|h\| \to 0$ and Ω_2, P_2, H_2, satisfy the conditions of the functions S_3, A_2, H_2 in system (3.14) of Theorem III.4. Since by a linear transformation, a may be taken in a neighborhood of zero, the above theorem for the case where the periodic solutions are not isolated follows from Theorem III.4. If the periodic solutions are isolated, then the result follows immediately from Theorem III.7 and III.9.

Remark V.16. The above theorem may obviously be generalized to systems where more than two vector variables have the same behavior as those in (5.34).

Example V.17. Consider the system of differential equations ($' = d/dt$)

$$
\begin{aligned}
x'' + f(x, x') &= \mu(t)\, F(x, x', y, y'), \\
y'' + g(y, y') &= \eta(t)\, G(x, x', y, y')
\end{aligned}
\tag{5.38}
$$

where $f, g, F, G \in C^2$, $\mu, \eta \in C^1$ and each of the equations $x'' + f(x, x') = 0$, $y'' + g(y, y') = 0$ has an isolated periodic solution $x = x^0(\omega t + \varphi)$, $y = y^0(\nu t + \varphi)$, respectively, and also each of these periodic solutions has a negative characteristic exponent. If $\mu(t) \to 0$, $\eta(t) \to 0$ as $t \to \infty$, then from Theorem V.15, the torus defined by $x = x^0(\omega t + \varphi)$, $y = y^0(\nu t + \varphi)$ is an asymptotically stable manifold of (5.38). If $\int\limits_0^\infty |\mu(t)|\, dt < +\infty$, $\int\limits_0^\infty |\eta(t)|\, dt < +\infty$, then this torus is asymptotically stable with asymptotic phase. This example generalizes the result in Example V.8 and in particular the result of L. Markus mentioned there.

Theorem V.18. *Consider a system of the form*

$$
\begin{aligned}
\frac{dx}{dt} &= X(x) + X^*(x, y, z), \\
\frac{dy}{dt} &= Y(y) + Y^*(x, y, z), \\
\frac{dz}{dt} &= P(x, y)\, z + Z(x, y, z)
\end{aligned}
\tag{5.39}
$$

where x, y, z are vectors; X, Y, P, X^*, Y^*, and $Z \in C^2$; $0 = X^*(x, y, 0) = Y^*(x, y, 0) = Z(x, y, 0) = \partial Z(x, y, 0)/\partial z$; each of the systems $dx/dt = X(x)$, $dy/dt = Y(y)$ has a manifold of solutions defined by $x = x^0[\omega(b)\, t + \varphi, b]$, $y = y^0[\nu(c)\, t + \varphi, c]$, respectively, where φ is a constant, b, c are constant vectors and each of these manifolds satisfies the conditions of Theorem V.1. If there exists a function $\sigma(b, c) > 0$ and continuous for all b, c such that the fundamental system $Z_1(t)$, $Z_1(0) = I$, of the system of equations

$$
\frac{dz}{dt} = P[x^0, y^0]\, z,
\tag{5.40}
$$

satisfies a relation of the form

$$
\|Z_1(t)\, Z_1^{-1}(u)\| \leqq K\, e^{-\sigma(b,c)\,(t-u)}, \qquad t \geqq u \geqq 0, \qquad K > 0,
\tag{5.41}
$$

then the manifold of solutions of (5.39) defined by $x = x^0[\omega(b)\, t + \varphi, b]$, $y = y^0[\nu(c)\, t + \varphi, c]$, $z = 0$ is asymptotically stable with asymptotic phase and amplitude.

Proof: As in the proof of Theorem V.15, the transformation $x = x^0(s_1, b) + Q_1(s_1, b)\, h_1$, $y = y^0(s_2, c) + Q_2(s_2, c)$, $z = z$ transforms system (5.39) into an equivalent system

$$
\begin{aligned}
\frac{ds}{dt} &= \mu(a) + \Omega_1(a, h, s) + \Omega_2(a, h, z, s), \\
\frac{da}{dt} &= P_1(a, h, s) + P_2(a, h, z, s), \\
\frac{dh}{dt} &= H(a)\, h + H_1(a, h, s) + H_2(a, h, z, s), \\
\frac{dz}{dt} &= P[x^0(s_1, b), \quad y^0(s_2, c)]\, z + Z(a, h, z, s),
\end{aligned}
$$

where $s=(s_1, s_2)$, $a=(b, c)$, $h=(h_1, h_2)$, $\mu(a)=[\omega(b), \nu(c)]$, $H(a)=\operatorname{diag}[H_1^*(b), H_2^*(c)]$ where the eigenvalues of $H_1^*(b)$, $H_2^*(c)$ have negative real parts for every b, c. The functions Ω_1, P_1, H_1 are $O(\|h\|^2)$; Ω_2, P_2, H_2 are $O(\|z\|)$ and Z is $O(\|z\|(\|z\|+\|h\|))$ as $\|h\| \to 0$, $\|z\| \to 0$. Also by a linear transformation, we may assume a to be in a neighborhood of zero. The conclusion of the theorem now follows from Theorem III.4.

Remark V.19. It is generally a very difficult problem to decide when condition (5.41) can be satisfied, since $P[x^0, y^0]$ is an almost periodic matrix. In case $P(x, y)$ were only a function of one of the variables x or y, then the problem would be completely solved by the Floquet theory. However, in the general case, only fragmentary results have been obtained (see, *e.g.*, [*7*, *15*]).

Remark V.20. Generalizations of systems (5.39) along the lines of the preceding theorem could be given. Also, one could suppose there were more than two vector variables which had the same behavior as x, y in (5.39).

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On the Singularities of Generalized Axially Symmetric Potentials

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I. Introduction

Recently G. SZEGÖ [*1*], Z. NEHARI [*2*], and A. ERDÉLYI [*3*] have obtained some interesting results connecting the singularities of axially-symmetric potentials with those of analytic functions. It has been shown by the author [*4*], [*5*] that a similar connection also exists between the singularities of a three-dimensional potential and a function of two complex variables. This was done by considering the Whittaker-Bergman operator [*6*], [*7*][*] $B_3(f, \mathscr{L}, X_0)$ which transforms functions of two complex variables $f(t, u)$ into three-dimensional harmonic functions, and by using a modified form of an idea previously employed by HADAMARD in the proof of his theorem on the multiplication of singularities [*2*], [*8*]. In this paper we shall use these means to investigate the singularities of solutions to the partial differential equation

$$L[u] \equiv \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial \varrho^2} + \frac{k}{\varrho} \frac{\partial u}{\partial \varrho} = 0 \tag{1}$$

(where $k \geqq 0$), referred to by WEINSTEIN [*9*] and ERDÉLYI [*3*] as the differential equation of generalized axially symmetric potentials. This equation is arrived at by considering those solutions of the n-dimensional Laplace equation, $\frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_n^2} = 0$, which depend solely on the variables $z = x_1$, $\varrho = (x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

Solutions to equation (1) may be constructed by a modified form of the Whittaker-Bergman operator. However, in this case the operator transforms functions of a single complex variable $f(\sigma)$ into axially symmetric potentials $u(r, \vartheta)\left(r^2 = z^2 + \varrho^2, \text{ and } \vartheta = \cos^{-1} \frac{z}{r}\right)$, that is

$$u(r, \vartheta) = A_n(f, \mathscr{L}, X_0) \equiv \int_{\substack{+1 \\ \mathscr{L}}}^{-1} f(\sigma) \left(u - \frac{1}{u}\right)^{k-1} \frac{du}{u} \tag{2}$$

---

[*] $$H(X) = B_3(f, \mathscr{L}, X_0), \quad B_3(f, \mathscr{L}, X_0) = \frac{1}{2\pi i} \int_{\mathscr{L}} f(t, u) \frac{du}{u},$$

$$t = \left[-(x - iy)\frac{u}{2} + z + (x + iy)\frac{u^{-1}}{2}\right],$$

$|X_0 - X| < \varepsilon$, $X \equiv (x, y, z)$, $X_0 \equiv (x_0, y_0, z_0)$, where $\mathscr{L}$ is a differentiable arc in the u-plane, and $\varepsilon > 0$ is sufficiently small.

where $\sigma = z + \frac{i}{2}\varrho\left(u + \frac{1}{u}\right)$, $|X - X^0| < \varepsilon$, $X \equiv (x_1, x_2, \ldots, x_n)$, $X^0 \equiv (x_1^0, x_2^0, \ldots, x_n^0)$, $\varepsilon > 0$ is sufficiently small, and the integration path is the upper semi-circular arc*. Now, if $f(\sigma)$ has a Taylor expansion convergent in some neighborhood of $\sigma = 0$, that is

$$f(\sigma) = \sum_{n=0}^{\infty} a_n \sigma^n, \tag{3}$$

then the operator $A_n(f, \mathscr{L}, X_0)$ defines an axially symmetric potential

$$u(r, \vartheta) = \frac{(4i)^k (\Gamma_{(k/2)})^2}{4} \sum_{n=0}^{\infty} a_n \frac{n!}{\Gamma(k-n)} r^n C_n^{k/2}(\cos\vartheta) \tag{4}$$

for r sufficiently small. This may be seen by considering the classical identity [*10*]

$$r^n C_n^{\lambda}(\cos\vartheta) = \frac{2^{1-2\lambda}}{n!} \frac{\Gamma(2\lambda+n)}{[\Gamma(\lambda)]^2} \int_0^{\pi} [z + i\varrho\cos\varphi]^n (\sin\varphi)^{2\lambda-1} d\varphi. \tag{5}$$

Occasionally it is convenient to continue the arguments of $u(r, \vartheta)$ to complex values, for instance if we introduce the particular continuation

$$r = +(z^2 + \varrho^2)^{\frac{1}{2}},$$

$$\xi = \frac{z}{r},$$

which reduces to $\xi = \cos\vartheta$ for real z, ϱ, we may obtain an inverse operator.

Lemma. *Let $u(r, \vartheta)$ be an axially symmetric potential regular at the origin, that is*

$$u(r, \vartheta) = \int_{+1}^{-1} f(\sigma)\left(u - \frac{1}{u}\right)^{k-1} \frac{du}{u},$$

where

$$|u| = 1, \quad \operatorname{Im} u \geqq 0$$

$$f(\sigma) = \sum^{\infty} a_n \sigma^n;$$

then $f(\sigma)$ may be generated by the following integral transform

$$f(\sigma) = \frac{k}{(2i)^k \pi} \int_{\substack{-1 \\ C}}^{+1} W(r, \xi) \frac{(1-\xi^2)^{\frac{k-1}{2}}\left(1 - \frac{\sigma^2}{r^2}\right)}{\left(1 - 2\xi\frac{\sigma}{r} + \frac{\sigma^2}{r^2}\right)^{\frac{k}{2}-1}} d\xi, \tag{6}$$

where $W(r, \cos\vartheta) \equiv u(r, \vartheta)$, and C is the real axis.

Proof. Let us define

$$K\left(\frac{\sigma}{r}, \xi\right) = \left[\frac{\Gamma(k)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k}{2}\right)}\right]^2 (1-\xi^2)^{\frac{k-1}{2}} \frac{2^{2-3k}}{i^k} \sum_{m=0}^{\infty} (2m+k)\left(\frac{\sigma}{r}\right)^m C_m^{k/2}(\xi); \tag{7}$$

---

* Our integrand is an analytic function of ϱ; hence our integral is a Cauchy integral, and we need not restrict our path of integration to the upper semi-circular arc.

it may then be readily seen from the orthogonality relation for the Gegenbauer polynomials

$$\int_{\substack{-1\\C}}^{+1}(1-\xi^2)^{\frac{k}{2}-1}C_n^{k/2}(\xi)\,C_m^{k/2}(\xi)\,d\xi=\frac{2^k\,\Gamma(k+n)}{(2n+k)\,n!}\left[\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k)}\right]^2\delta_{nm} \tag{8}$$

(where C is the real axis) that

$$f(\sigma)=\sum_{n=0}^{\infty}a_n\sigma^n=\int_{-1}^{+1}K\left(\frac{\sigma}{r},\xi\right)\left\{a\sum_{m=0}^{\infty}\frac{a_m\,m!}{\Gamma(m+k)}\,r^m\,C_m^{k/2}(\xi)\right\}d\xi,$$

(where $a=(4i)^k\left[\Gamma\left(\frac{k}{2}\right)\right]^2/4$). We note that $K\left(\frac{\sigma}{r},\xi\right)$ may be formally summed to give

$$\begin{aligned}K\left(\frac{\sigma}{r},\xi\right)&=\frac{2^{2-3k}}{i^k}\left[\frac{\Gamma(k)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}\right]^2(1-\xi^2)^{\frac{k-1}{2}}\left\{t^{1-k}\frac{\partial}{\partial t}\left[t^k\sum_{n=0}^{\infty}t^{2n}\,C_n^{k/2}(\xi)\right]\right\}_{t=\sqrt{\frac{\sigma}{r}}}\\&=\frac{k\,2^{2-3k}}{i^k}\left[\frac{\Gamma(k)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}\right]^2\frac{(1-\xi^2)^{\frac{k-1}{2}}\left(1-\frac{\sigma^2}{r^2}\right)}{\left(1-2\xi\frac{\sigma}{r}+\frac{\sigma^2}{r^2}\right)^{\frac{k}{2}-1}},\end{aligned} \tag{9}$$

providing $\left|\frac{\sigma}{r}\right|<1$; this follows from the classical identity

$$\sum_{n=0}^{\infty}t^n\,C_n^{k/2}(\xi)=(1-2\xi t+t^2)^{-k/2},\qquad\text{for}\qquad|t|<1.$$

The constant coefficient may be reduced by using LEGENDRE's duplication formula,

$$\frac{\Gamma(2z)}{\Gamma(z)}=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma\left(z+\frac{1}{2}\right);$$

one then obtains for the kernel

$$K\left(\frac{\sigma}{r},\xi\right)=\frac{k}{(2i)^k\,\pi}\,\frac{(1-\xi^2)^{\frac{k-1}{2}}\left(1-\frac{\sigma^2}{r^2}\right)}{\left(1-2\xi\frac{\sigma}{r}+\frac{\sigma^2}{r^2}\right)^{\frac{k}{2}-1}}.$$

We now note that $K\left(\frac{\sigma}{r},\xi\right)$ is analytic in ξ, and also, since $u(r,\vartheta)$ is harmonic $W(r,\xi)$ is analytic in ξ except for the points $\xi=\pm1$. We may conclude from this that the integral involved in our representation for $f(\sigma)$ is a Cauchy-integral and that we need not restrict our integration path to the real axis.

II. Singularities of Generalized Axially Symmetric Potentials

In a previous paper [4], the author proved the following theorems concerning the singularities of three-dimensional harmonic functions.

Theorem 1. *If* $Z^3=E\{S(X|u)=0\}$, *is the singularity manifold of* $\frac{1}{u}f(t,u)$, *then*

$$H(X)=\frac{1}{2\pi i}\int_{\mathscr{L}}f(t,u)\,\frac{du}{u}$$

(where $\mathscr{L}$ is the unit circle) is regular at $X=(x, y, z)$ providing this point does not lie simultaneously on the two surfaces $S(X|u)=0$, and $\frac{\partial}{\partial u} S(X|u)=0$.

Theorem 2. *Let $H^{\infty}(X)$ be a harmonic function regular at infinity* $\left(|X|>\frac{1}{\varepsilon}\right)$, and let $V(r, \xi, \varrho)$ be the function obtained from $H^{\infty}(X)$ by replacing x, y, z by*

$$x=\frac{r}{2}(\varrho+\varrho^{-1})\sqrt{1-\xi^2},$$

$$y=\frac{r}{2i}(\varrho-\varrho^{-1})\sqrt{1-\xi^2},$$

$$z=r\xi.$$

Furthermore, let $Z\equiv E^5\{r=\Phi(\xi, \varrho)\}$ be the singularity manifold of $V(r, \xi, \varrho)=H^{\infty}(X)$, $X\in ¢^3$ (where $¢^3$ is a region in the ideal, Euclidean three-space, that is the three-dimensional space of complex points), then the functions of two complex variables $g(s, u)$ which may be generated by the integral operator

$$g(s, u)=\frac{1}{4\pi i}\int_{-1}^{+1}\left[\int_{\Gamma}\frac{r(s+t)}{(s-t)^2}V(r, \xi, \varrho)\frac{d\varrho}{\varrho}\right]d\xi$$

(the integration path in the ξ-plane is the real axis connecting -1 to $+1$, and the path in the ϱ-plane is the unit circle) is regular at (s, u) providing (s, u) does not lie on the "envelope" of the three-parameter family

$$\psi(s, u|r, \xi, \varrho)\equiv t-s=r\left[\xi+\frac{i}{2}\sqrt{1-\xi^2}\left(\frac{u}{\varrho}+\frac{\varrho}{u}\right)\right]-S=0$$

subject to the auxillary condition $r=\Phi(\xi, \varrho)$.

We now state the following two theorems concerning the singularities of generalized axially symmetric potentials without proof, since their proofs are essentially identical in structure to those of Theorems 1 and 2. We mention in passing, however, that the proof of these theorems depends solely on the concept of attempting to enlarge the original domain of definition for a function given in terms of an integral representation by continuously deforming the integration path, subject to the condition that in this process of deformation the integration path at no time crosses a singularity of the integrand.

Theorem 3. *Let $Z^2\equiv E\{S(r, \vartheta|u)=0\}$ be the singularity manifold of $\frac{1}{u}f(\sigma)\times\left(u-\frac{1}{u}\right)^{k-1}$; then the axially-symmetric potential*

$$u(r, \vartheta)=A_n(f, \mathscr{L}, X_0)\equiv\int_{+1}^{-1}f(\sigma)\left(u-\frac{1}{u}\right)^{k-1}\frac{du}{u}$$

$$(|u|=1, \quad \operatorname{Im} u\geqq 0)$$

is regular at (r, ϑ) providing (r, ϑ) does not lie simultaneously on the two surfaces

$$S(r, \vartheta|u)=0, \quad \text{and} \quad \frac{\partial}{\partial u}S(r, \vartheta|u)=0.$$

---

* $H^{\infty}(X)=\frac{1}{2\pi i}\int_{\mathscr{L}}g(t, u)\frac{du}{u}$, and $g(t, u)$ has the form $g(t, u)=\sum_{n=0}^{\infty}\sum_{m=-0}^{+n}a_{mn}t^{-n-1}u^m$.

Theorem 4. *Let* $u(r, \vartheta)$ *be a generalized axially symmetric potential, and let* $W(r, \xi)$ *be a function obtained from* $u(r, \vartheta)$ *by replacing* ϑ *by* $\cos^{-1}\left(\frac{z}{r}\right)$. *Furthermore, let* $Z \equiv E^3[r = \chi(\xi)\}$ *be the singularity manifold of* $W(r, \xi)$; *then the function*

$$f(\sigma) = \frac{k}{\pi(2i)^k} \int\limits_{\substack{-1\\ C}}^{+1} W(r, \xi) \frac{(1-\xi^2)^{\frac{k-1}{2}}\left(1-\frac{\sigma^2}{r^2}\right)}{\left(1-2\xi\frac{\sigma}{r}+\frac{\sigma^2}{r^2}\right)^{\frac{k}{2}-1}}\, d\xi$$

(*C is the real axis between* -1 *and* $+1$) *is regular at* σ *providing this point does not lie on the "envelope" of the two parameter family* $\psi(\sigma | r, \xi) \equiv r^2 - 2\sigma\xi + \sigma^2 = 0$, *subject to the auxillary condition* $r = \chi(\xi)$. *This may be restated by saying* σ *must not lie simultaneously on the two curves*

$$\psi_1(\sigma | \xi) \equiv \chi^2(\xi) - 2\sigma\xi\chi(\xi) + \sigma^2 = 0,$$

and $\frac{\partial \psi_1}{\partial \xi} = 0$.

To illustrate the use of Theorems 3 and 4, we consider the case where $\sigma = \alpha$ is a singular point of $\frac{1}{u} f(\sigma)\left(u - \frac{1}{u}\right)^{k-1}$. Here the singularities of the integrand occur for $\sigma \equiv z + \frac{i}{2}\varrho\left(u + \frac{1}{u}\right) = \alpha$, and we may represent the singularity manifold as

$$S(z, \varrho | u) \equiv u(z - \alpha) + \frac{i}{2}\varrho(u^2 + 1) = 0.$$

Eliminating u between $S = 0$ and $\frac{\partial S}{\partial u} = 0$, we obtain the locus $\varrho^2 + (z - \alpha)^2 = 0$. If $u(r, \vartheta)$ is regular in the unit sphere but has singularities on $r = 1$, they are located at $z = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)$. When $k = 1$, $C_n^{\frac{1}{2}}(\xi) = P_n(\xi)$, and this is equivalent to NEHARI's result concerning Legendre expansions [2].

We next consider the possible singularities of an $f(\sigma)$ generated from a $W(r, \xi)$ with the singularities $\varrho^2 + (\alpha - z)^2 = r^2 - 2\alpha\xi r + \alpha^2 = 0$. This is done by obtaining the "envelope" of the two parameter family $\psi(\sigma | r, \xi) \equiv r^2 - 2\sigma\xi + \sigma^2 = 0$ subject to $\xi = \frac{1}{2}\left(\frac{r}{\alpha} + \frac{\alpha}{r}\right)$. Eliminating r between $\psi_2 \equiv r^2 - \sigma\left(\frac{r^2}{\alpha} + \alpha\right) + \sigma^2 = 0$, and $\frac{\partial \psi_2}{\partial r} \equiv 2r\left(1 - \frac{\sigma}{\alpha}\right) = 0$ yields $\sigma = \alpha$. We shall now attempt to formulate a necessary and sufficient criteria for the occurence of singularities from the above results.

The points $\xi = \pm 1$, and $u = \pm 1$, because they are the terminal points for the paths of integration, have a special status; indeed, a branch of $W(r, \xi)$ may have a singularity at $\xi = \pm 1$, which does not correspond to $f(\sigma)$ at $\sigma = \pm r$. $f(\sigma)$ on the other hand will not be singular at $u = \pm 1$ for all X, hence all the singularities of $f(\sigma)$ will correspond to singularities of $W(r, \xi)$. We realize, therefore, that $W(r, \xi)$ is singular if and only if the point (r, ξ) lies on the locus $r^2 - 2\alpha r\xi + \alpha^2 = 0$ (here $\sigma = \alpha$ is a singularity of $f(\sigma)$), the one exception being points on the z-axis $(r, \pm 1)$, which may be singular points of $W(r, \xi)$ without corresponding to singularities of $f(\sigma)$.

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